Q1. $\boxed{\gcd(2217, 117) = d}$

$2217 = 18 \cdot 117 + 111$

$117 = 1 \cdot 111 + 6$

$111 = 18 \cdot 6 + 3$

$6 = 2 \cdot 3 + 0$ $\Rightarrow$ $d = 3$ as $\gcd(2217, 117) = 3$

2

(ii) $3 = 111 - 18 \cdot 6$

$= 111 - 18 \cdot [117 - 1 \cdot 111]$

$= 111 - 18 \cdot 117 + 18 \cdot 111$

$= 19 \cdot 111 - 18 \cdot 117$

2

$= 19 \cdot [2217 - 18 \cdot 117] - 18 \cdot 117$

$= 19 \cdot 2217 - 342 \cdot 117 - 18 \cdot 117$

$\therefore 3 = 19 \cdot 2217 - 360 \cdot 117$

$\boxed{3 = 2217r + 117s \Rightarrow r = 19 \ \& \ S = -360}$

Q1.(b) $433_6 \Rightarrow$

6^2	6^1	6^0
4	3	3

$433_6 = 3 \times 6^0 + 3 \times 6^1 + 4 \times 6^2$

$= 3 \times 1 + 3 \times 6 + 4 \times 36$

$= 3 + 6 + 144$

$\therefore 433_6 = 165$

2

$165 = 23 \cdot 7 + 4$

$23 = 3 \cdot 7 + 2$

$3 = 0 \cdot 7 + 3$

$\boxed{\therefore 433_6 = 324_7}$

Q2. $$x_{n+1} = 2\left(\frac{-3x_n + 8}{x_n}\right)$$

3

$$\lambda = 2\left(\frac{-3\lambda + 8}{\lambda}\right)$$

$$\lambda^2 = 2(-3\lambda + 8)$$

$$\lambda^2 = -6\lambda + 16$$

$$\lambda^2 + 6\lambda - 16 = 0$$

$$(\lambda + 8)(\lambda - 2) = 0$$

$$\lambda = -8 \;\&\; \lambda = 2$$

Let iteration, $\boxed{x_n = \lambda}$ ($x_{n-1}, x_n, x_{n+1}, x_{n+2}, \ldots$ should all converge to same value to ≥ 6 dps … so can simplify algebraic expression by using λ to find fixed point)

$\therefore$ 2 fixed points when $x = -8$ and $x = 2$

(May use λ or x to solve)

Q3. $W = 2U - 3V$

a)

$$W = 2\begin{pmatrix} 2 & 0 & (\lambda-1) \\ 0 & 4 & \lambda \\ 1 & -1 & (2\lambda+2) \end{pmatrix} - 3\begin{pmatrix} 1 & -1 & 0 \\ -1 & 3 & 0 \\ 0 & -1 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} 4 & 0 & (2\lambda-2) \\ 0 & 8 & 2\lambda \\ 2 & -2 & (4\lambda+4) \end{pmatrix} + \begin{pmatrix} -3 & 3 & 0 \\ 3 & -9 & 0 \\ 0 & 3 & -9 \end{pmatrix}$$

$$\therefore \quad W = \begin{pmatrix} 1 & 3 & (2\lambda-2) \\ 3 & -1 & 2\lambda \\ 2 & 1 & (4\lambda-5) \end{pmatrix}$$

1

b)

$$\text{Det}\, W = 1\begin{vmatrix} -1 & 2\lambda \\ 1 & (4\lambda-5) \end{vmatrix} - 3\begin{vmatrix} 3 & 2\lambda \\ 2 & (4\lambda-5) \end{vmatrix} + (2\lambda-2)\begin{vmatrix} 3 & -1 \\ 2 & 1 \end{vmatrix}$$

$$= 1\left(-(4\lambda-5) - 2\lambda\right) - 3\left(3(4\lambda-5) - 4\lambda\right) + (2\lambda-2)\left(3-(-2)\right)$$

$$= (-4\lambda + 5 - 2\lambda) - 3(12\lambda - 15 - 4\lambda) + (2\lambda-2)(5)$$

$$= -6\lambda + 5 - 24\lambda + 45 + 10\lambda - 10$$

$$\therefore \quad \text{Det}\, W = -20\lambda + 40$$

$\boxed{\text{If Singular} \Rightarrow \text{Det}\, W = 0}$

4

$$\therefore \quad -20\lambda + 40 = 0$$

$$-20\lambda = -40$$

$$\lambda = 2$$

Q4.

$$L: \frac{x-3}{2} = \frac{y-13}{4} = \frac{z-2}{1} \quad (=\lambda) \Rightarrow \boxed{\begin{aligned} x &= 3+2\lambda \\ y &= 13+4\lambda \\ z &= 2+\lambda \end{aligned}}$$

Change Symmetric → Parametric Form

$$M: \frac{x+5}{1} = \frac{y-3}{-1} = \frac{z-3}{-2} \quad (=\mu) \Rightarrow \boxed{\begin{aligned} x &= -5+\mu \\ y &= 3-\mu \\ z &= 3-2\mu \end{aligned}}$$

Let $x = x$: $3+2\lambda = -5+\mu$

$\therefore 8+2\lambda = \mu$

$\mu = (8+2\lambda)$

Let $y = y$: $13+4\lambda = 3-\mu$

$13+4\lambda = 3-(8+2\lambda)$

$13+4\lambda = 3-8-2\lambda$

$6\lambda = -18$

$\therefore \lambda = -3$ & $\mu = 8+2\lambda$
$= 8-6$

$\therefore \mu = 2$

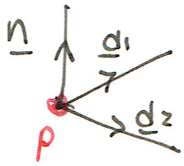
$\boxed{\lambda = -3}$ $\boxed{\mu = 2}$

L	M
$x = 3+2\lambda = 3-6 = -3$	$x = -5+\mu = -5+2 = -3$
$y = 13+4\lambda = 13-12 = 1$	$y = 3-\mu = 3-2 = 1$
$z = 2+\lambda = 2-3 = -1$	$z = 3-2\mu = 3-2(2) = -1$

$\therefore$ Lines L & M intersect at $P(-3, 1, -1)$ (4)

Q4. $L: \frac{x-3}{2} = \frac{y-13}{4} = \frac{z-2}{1} \Rightarrow d_1 = (2,4,1)$

$M: \frac{x+5}{1} = \frac{y-3}{-1} = \frac{z-3}{-2} \Rightarrow d_2 = (1,-1,-2)$



Equation of a Plane $\boxed{r.n = a.n}$

$$\underline{n} = \underline{d}_1 \times \underline{d}_2 = \begin{pmatrix} \underline{i} & \underline{j} & \underline{k} \\ 2 & 4 & 1 \\ 1 & -1 & -2 \end{pmatrix}$$

$$\therefore \underline{n} = \underline{i}\begin{vmatrix} 4 & 1 \\ -1 & -2 \end{vmatrix} - \underline{j}\begin{vmatrix} 2 & 1 \\ 1 & -2 \end{vmatrix} + \underline{k}\begin{vmatrix} 2 & 4 \\ 1 & -1 \end{vmatrix}$$

$$= \underline{i}(-8-(-1)) - \underline{j}(-4-1) + \underline{k}(-2-4)$$

$$\Rightarrow n = -7\underline{i} + 5\underline{j} - 6\underline{k}$$

3

& $P(-3, 1, -1)$ from (a)

$\boxed{r.n = a.n}$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} -7 \\ 5 \\ -6 \end{pmatrix} = \begin{pmatrix} -3 \\ 1 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} -7 \\ 5 \\ -6 \end{pmatrix}$$

$$-7x + 5y - 6z = 21 + 5 + 6 = 32$$

$$\therefore \pi_1 : -7x + 5y - 6z = 32 \quad (\text{OR } 7x - 5y + 6z = -32)$$

c) $\underline{n} = (-7, 5, -6)$ & if $\pi_2 = 5x + y - 2z = 10 \Rightarrow \underline{m} = (5, 1, -2)$

$$\boxed{\cos\theta = \frac{\underline{n}\cdot\underline{m}}{|\underline{n}||\underline{m}|}} = \frac{\begin{pmatrix} -7 \\ 5 \\ -6 \end{pmatrix} \cdot \begin{pmatrix} 5 \\ 1 \\ -2 \end{pmatrix}}{\sqrt{49+25+36}\sqrt{25+1+4}} = \frac{-35+5+12}{\sqrt{110}\sqrt{30}} = \frac{-18}{\sqrt{3300}}$$

$$\therefore \theta = \cos^{-1}(-0.3133..)$$

$$\theta = 108.3$$

$$\therefore \text{Acute Angle} = (180 - 108.3) = 71.7^\circ$$

3

Q5.

$f(x) = e^x \quad f(0) = e^0 = 1$

$f'(x) = e^x \quad f'(0) = 1$

$f''(x) = e^x \quad f''(0) = 1$

$f'''(x) = e^x \quad f'''(0) = 1$

$f(x) \approx f(0) + f'(0)\frac{x}{1!} + f''(0)\frac{x^2}{2!} + \dots + f^n(0)\frac{x^n}{n!}$

$\therefore e^x \simeq 1 + (1)\frac{x}{1!} + (1)\frac{x^2}{2!} + (1)\frac{x^3}{3!} + \dots$

$\Rightarrow e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6}$ (to x^3)

1

Let $g(x) = \ln\left(\frac{1}{1-x}\right) = \ln(1) - \ln(1-x) = -\ln(1-x)$

Then $g(x) = -\ln(1-x)$

$g'(x) = -\frac{1}{1-x} \cdot -1 = \frac{1}{(1-x)} = (1-x)^{-1}$

$g''(x) = -(1-x)^{-2} \cdot -1 = (1-x)^{-2}$

$g'''(x) = -2(1-x)^{-3} \cdot -1 = 2(1-x)^{-3}$

$g(0) = -\ln(1) = 0$

$g'(0) = \frac{1}{(1-0)} = 1$

$g''(0) = \frac{1}{(1-0)^2} = 1$

$g'''(0) = \frac{2}{(1-0)^3} = 2$

$\therefore \ln\left(\frac{1}{1-x}\right) = 0 + (1)\frac{x}{1!} + (1)\frac{x^2}{2!} + (2)\frac{x^3}{3!} + \dots$

$= x + \frac{x^2}{2} + \frac{x^3}{3}$ (to x^3)

3

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c) $\ln\left(\frac{e^{e^{-x}}}{1-x}\right) = \ln\left(e^{e^{-x}}\right) - \ln(1-x)$

$= e^{-x} + \ln(1-x)^{-1}$

$= e^{-x} + \ln\left(\frac{1}{1-x}\right)$

$= \left[1 + (-x) + \frac{(-x)^2}{2} + \frac{(-x)^3}{6}\right] + \left[x + \frac{x^2}{2} + \frac{x^3}{3}\right]$

$= 1 \cancel{-x} + \frac{x^2}{2} - \frac{x^3}{6} \cancel{+x} + \frac{x^2}{2} + \frac{x^3}{3}$

2

$= 1 + x^2 - \frac{x^3}{6} + \frac{2x^3}{6}$

$\therefore \ln\left(\frac{e^{e^{-x}}}{1-x}\right) = 1 + x^2 + \frac{x^3}{6}$

Q6.

$$\boxed{9\frac{d^2y}{dx^2} - 12\frac{dy}{dx} + 4y = 4x^2 - 2}$$

Aux. Eqn $\quad 9m^2 - 12m + 4 = 0$

$$(3m-2)^2 = 0$$

$$m = 2/3 \text{ (twice)}$$

Complementary Function $\quad y_c = Ae^{2/3x} + Bxe^{2/3x}$

Particular Integral

Let $y_p = ax^2 + bx + c$; $\frac{dy_p}{dx} = 2ax + b$ & $\frac{d^2y_p}{dx^2} = 2a$

$$9\frac{d^2y}{dx^2} - 12\frac{dy}{dx} + 4y = 4x^2 - 2$$

$$9(2a) - 12(2ax+b) + 4(ax^2+bx+c) = 4x^2 - 2$$

$$18a - 24ax - 12b + 4ax^2 + 4bx + 4c = 4x^2 - 2$$

$$4ax^2 + (4b - 24a)x + (18a - 12b + 4c) = 4x^2 - 2$$

Then $4ax^2 = 4x^2$ & $(4b - 24a) = 0$ & $18a - 12b + 4c = -2$

$\therefore \underline{\underline{a = 1}}$

$4b - 24 = 0$

$4b = 24$

$\therefore \underline{\underline{b = 6}}$

$18 - 72 + 4c = -2$

$4c = 52$

$\therefore \underline{\underline{c = 13}}$

So $y_p = ax^2 + bx + c$

Becomes $y_p = x^2 + 6x + 13$ $\quad$ So General Solution is

$$y = y_c + y_p \Rightarrow \boxed{y = Ae^{2/3x} + Bxe^{2/3x} + x^2 + 6x + 13}$$

$$\boxed{y = Ae^{2/3x} + Bxe^{2/3x} + x^2 + 6x + 13}$$

$\boxed{\begin{array}{l} x=0 \\ y=12 \end{array}}$

$12 = Ae^{0} + 0 + 0 + 0 + 13$

$12 = A + 13$

$\Rightarrow \underline{A = -1}$

If $y = Ae^{2/3x} + Bxe^{2/3x} + x^2 + 6x + 13$

then $\boxed{\dfrac{dy}{dx} = \dfrac{2}{3}Ae^{2/3x} + \left(Be^{2/3x} + \dfrac{2}{3}Bxe^{2/3x}\right) + 2x + 6}$

(Product Rule)

$\boxed{\begin{array}{l} x=0 \\ \dfrac{dy}{dx} = \dfrac{28}{3} \end{array}}$

$\dfrac{28}{3} = \dfrac{2}{3}Ae^{0} + Be^{0} + 0 + 0 + 6$

$\dfrac{28}{3} = -\dfrac{2}{3} + B + 6$

$28 = -2 + 3B + 18$

$28 = 16 + 3B$

$\therefore 3B = 12$

$\Rightarrow \underline{B = 4}$

Need $B=4$ + Particular Solution for this mark.

10

So Particular Solution is:

$$\boxed{y = -e^{2/3x} + 4xe^{2/3x} + x^2 + 6x + 13}$$

Q7 Let $n=1$

$$A = \begin{pmatrix} 1 & (1)p & \left((1)q + \frac{pr}{2}(1-1)\right) \\ 0 & 1 & (1)r \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & p & q \\ 0 & 1 & r \\ 0 & 0 & 1 \end{pmatrix}$$

✓ true for $n=1$

Assume true for $n=k$

$$A^k = \begin{pmatrix} 1 & kp & \left(kq + \frac{kpr(k-1)}{2}\right) \\ 0 & 1 & kr \\ 0 & 0 & 1 \end{pmatrix}$$

Consider $n=k+1$

$$A^{k+1} = A^k . A^1 = \begin{pmatrix} 1 & kp & \left(kq + \frac{kpr(k-1)}{2}\right) \\ 0 & 1 & kr \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & p & q \\ 0 & 1 & r \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & p+kp & q + kpr + \left(kq + \frac{kpr(k-1)}{2}\right) \\ 0 & 1 & r+kr \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & (k+1)p & \left(\frac{2q + 2kpr + 2kq + k^2pr - kpr}{2}\right) \\ 0 & 1 & (k+1)r \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & (k+1)p & \left(\frac{kpr + 2q + 2kq + k^2pr}{2}\right) \\ 0 & 1 & (k+1)r \\ 0 & 0 & 1 \end{pmatrix}$$

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Q7 (continued)

$$A^{k+1} = \begin{pmatrix} 1 & (k+1)p & \left(\frac{kpr + 2q + 2kq + k^2pr}{2}\right) \\ 0 & 1 & (k+1)r \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & (k+1)p & \left(\frac{2q+2kq}{2} + \frac{k^2pr + kpr}{2}\right) \\ 0 & 1 & (k+1)r \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & (k+1)p & \left(\frac{\cancel{2}q(1+k)}{\cancel{2}} + \frac{kpr(k+1)}{2}\right) \\ 0 & 1 & (k+1)r \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & (k+1)p & \left((k+1)q + \frac{(k+1)pr((k+1)-1)}{2}\right) \\ 0 & 1 & (k+1)r \\ 0 & 0 & 1 \end{pmatrix}$$

If let $N = (k+1) \Rightarrow \begin{pmatrix} 1 & Np & \left(Nq + \frac{Npr(N-1)}{2}\right) \\ 0 & 1 & Nr \\ 0 & 0 & 1 \end{pmatrix}$

Not nec. but may help you see connection

6

As true for $n=1$, also assumed true for $n=k$ and by Proof of Mathematical Induction also true for $n=k+1$. Conjecture is true $\forall n \in \mathbb{N}$.