1. $P = \begin{pmatrix} 3 & 1 & -1 \\ 2 & 3 & 0 \\ -1 & -2 & 5 \end{pmatrix}$ & $kI = k\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} k & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & k \end{pmatrix}$

$$\therefore \quad P - kI = \begin{pmatrix} 3 & 1 & -1 \\ 2 & 3 & 0 \\ -1 & -2 & 5 \end{pmatrix} - \begin{pmatrix} k & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & k \end{pmatrix}$$

$$= \begin{pmatrix} 3-k & 1 & -1 \\ 2 & 3-k & 0 \\ -1 & -2 & 5-k \end{pmatrix}$$

If singular $\text{Det}(P-kI) = 0$

$$\text{Det} = (3-k)\begin{vmatrix} 3-k & 0 \\ -2 & 5-k \end{vmatrix} - 1\begin{vmatrix} 2 & 0 \\ -1 & 5-k \end{vmatrix} - 1\begin{vmatrix} 2 & 3-k \\ -1 & -2 \end{vmatrix}$$

$$= (3-k)\big((3-k)(5-k)-0\big) - 1\big(2(5-k)-0\big) - 1\big(-4+1.(3-k)\big)$$

$$= (3-k)(15-8k+k^2) - 2(5-k) + 4 - (3-k)$$

$$= 45 - 24k + 3k^2 - 15k + 8k^2 - k^3 - 10 + 2k + 4 - 3 + k$$

$$= -k^3 + 11k^2 - 36k + 36$$

6

As Singular $\Rightarrow$ Det $= 0$:

$$-k^3 + 11k^2 - 36k + 36 = 0$$

$$-(k-2)(k-6)(k-3) = 0$$

$$k=2;\ k=6;\ k=3$$

$$\begin{array}{r|rrrr} 3 & -1 & 11 & -36 & 36 \\ & \downarrow & -3 & 24 & -36 \\ \hline & -1 & 8 & -12 & \underline{\underline{0}} \end{array}$$

$-k^2 + 8k - 12$

$= -(k^2 - 8k + 12)$

$= -(k-2)(k-6)$

& As $R = 0 \Rightarrow (k-3)$ is a factor

Q2. $\boxed{\gcd(568, 26)}$

$$568 = 21.26 + 22$$
$$26 = 1.22 + 4$$
$$22 = 5.4 + 2$$
$$4 = 2.\underline{\underline{2}} + 0$$

$\therefore \gcd(568, 26) = 2$

$$\begin{aligned}\rightarrow 2 &= 22 - 5.4 \\ &= 22 - 5.(26 - 1.22) \\ &= 22 - 5.26 + 5.22 \\ &= 6.22 - 5.26 \\ &= 6.(568 - 21.26) - 5.26 \\ &= 6.568 - 126.26 - 5.26 \\ &= 6.568 - 131.26\end{aligned}$$

Thus $568r + 26s = 2$

So $568.(6) + 26.(-131) = 2 \Rightarrow r = 6$ & $s = -131$

4

Q3. $\boxed{\text{Prove by Induction } z^n = \cos n\theta + i\sin n\theta \quad \forall n \in \mathbb{Z}^+}$

If $z = \cos\theta + i\sin\theta$

Let $n=1$ $\quad z^1 = (\cos\theta + i\sin\theta)^1$

$\cos\theta + i\sin\theta = \cos(1\theta) + i\sin(1\theta)$ ✓ true for $n=1$

LHS = RHS

Assume true for $n=k$ $\quad z^k = \cos(k\theta) + i\sin(k\theta)$

Consider $n=k+1$

5

$$z^{k+1} = z.z^k$$

$$= (\cos\theta + i\sin\theta)(\cos(k\theta) + i\sin(k\theta))$$

$$= \cos\theta\cos(k\theta) + i\sin(k\theta)\cos\theta + i\sin\theta\cos(k\theta) + i^2\sin\theta\sin(k\theta)$$

Must collect Real & Imaginary separately

$$= [\underbrace{\cos\theta\cos(k\theta) - \sin\theta\sin(k\theta)}_{\text{Real}}] + i[\underbrace{\sin(k\theta)\cos\theta + \cos(k\theta)\sin\theta}_{\text{Imaginary}}]$$

$$= \cos(\theta + k\theta) + i\sin(k\theta + \theta)$$

$$= \cos(\theta(1+k)) + i\sin(\theta(k+1))$$

$$\therefore \quad z^{k+1} = \cos((1+k)\theta) + i\sin((k+1)\theta)$$

as required.

Thus as true for $n=1$, assumed true for $n=k$ and by proof of mathematical induction also true for $n=k+1$. We can therefore state $z^n = \cos n\theta + i\sin n\theta \quad \forall n \in \mathbb{Z}^+$

$$f(x) = e^{\sin x}$$

$$f'(x) = \cos x\, e^{\sin x}$$

$u = \cos x \qquad v = e^{\sin x}$

$u' = -\sin x \qquad v' = \cos x\, e^{\sin x}$

$$f''(x) = -\sin x\, e^{\sin x} + \cos^2 x\, e^{\sin x} = e^{\sin x}(\cos^2 x - \sin x)$$

$$f'''(x) = \boxed{u'v + uv'}$$

$u = e^{\sin x} \qquad v = (\cos x)^2 - \sin x$

$u' = \cos x\, e^{\sin x} \qquad v' = 2\cos x \sin x - \cos x$

$$f'''(x) = e^{\sin x}(\cos^3 x - \sin x \cos x - 2\sin x \cos x - \cos x)$$

$$= e^{\sin x}((\cos x)^3 - 3\sin x \cos x - \cos x)$$

$$\therefore f'''(x) = e^{\sin x}\left((\cos x)^3 - \frac{3}{2}\sin 2x - \cos x\right)$$

$$f^{iv}(x) = \boxed{u'v + uv'}$$

$u = e^{\sin x} \qquad v = (\cos x)^3 - \frac{3}{2}\sin 2x - \cos x$

$u' = \cos x\, e^{\sin x} \qquad v' = 3(\cos x)^2 . -\sin x - 3\cos 2x + \sin x$

$$= e^{\sin x}\left(\cos^4 x - \tfrac{3}{2}\sin 2x \cos x - \cos^2 x - 3\cos^2 x \sin x - 3\cos 2x + \sin x\right)$$

$$\therefore f^{iv}(x) = e^{\sin x}\left(\cos^4 x - \cos^2 x - 3\cos^2 x \sin x - \frac{3}{2}\sin 2x \cos x - 3\cos 2x + \sin x\right)$$

$$f(0) = e^{\sin 0} = e^0 = \underline{\underline{1}}$$

$$f'(0) = 1 \times e^0 = \underline{\underline{1}}$$

$$f''(0) = e^0(1^2 - 0) = \underline{\underline{1}}$$

$$f'''(0) = e^0(1 - 0 - 1) = \underline{\underline{0}}$$

$$f^{iv}(0) = e^0(1 - 1 - 0 - 0 - 3 + 0) = \underline{\underline{-3}}$$

5

$$f(x) = e^{\sin x} \simeq f(0) + f'(0)\frac{x}{1!} + f''(0)\frac{x^2}{2!} + f'''(0)\frac{x^3}{3!} + f^{iv}(0)\frac{x^4}{4!} + \dots + f^{n}(0)\frac{x^n}{n!}$$

$$= 1 + x + \frac{x^2}{2} + 0 - \frac{3x^4}{24} + \dots$$

$$\therefore e^{\sin x} = 1 + x + \frac{x^2}{2} - \frac{x^4}{8} + \dots \quad (\text{to } x^4)$$

Q5. $\boxed{P^4 = P^2 - 2I}$

$$P^8 = (P^4)^2$$
$$= (P^2 - 2I)^2$$
$$= (P^2 - 2I)(P^2 - 2I)$$
$$= P^4 - 2IP^2 - 2IP^2 + 4I^2$$
$$= P^4 - 4IP^2 + 4I^2$$
$$= P^4 - 4P^2 + 4I$$
$$= (P^2 - 2I) - 4P^2 + 4I$$
$$= -3P^2 + 2I$$

i.e $u = -3$ & $v = 2$

$4IP^2 = 4P^2$
$4I^2 = 4I$

4

Q6. $\pi_1: \quad x + y + 2z = 5$

$\pi_2: \quad x - y + 4z = 3$

$\pi_1 + \pi_2: \quad 2x + 6z = 8$

$2x = 8 - 6z$

$x = 4 - 3z$

let $z = t \Rightarrow x = 4 - 3t$

$\pi_1: x + y + 2z = 5$

$\pi_2: x - y + 4z = 3$

$\pi_1 - \pi_2: \quad 2y - 2z = 2$

$2y = 2z + 2$

$y = z + 1$

$\Rightarrow y = t + 1$

$x = 4 - 3t$

$x - 4 = -3t$

$\therefore \dfrac{x-4}{-3} = t$

$y = 1 + t$

$y - 1 = t$

$\therefore \dfrac{y-1}{1} = t$

$z = t$

$z - 0 = t$

$\therefore \dfrac{z-0}{1} = t$

4

$\Rightarrow \dfrac{x-4}{-3} = \dfrac{y-1}{1} = \dfrac{z-0}{1}$ is Line of Intersection of π_1 & π_2 in Symmetric form

Q6) b)

$\pi_1 : x+y+2z=5$

$\pi_2 : x-y+4z=3$

& $L: \frac{x-4}{-3} = \frac{y-1}{1} = \frac{z-0}{1}$

Symmetric: $\frac{x-4}{-3} = \frac{y-1}{1} = \frac{z-0}{1} = t$

Parametric ⇒ $x = -3t+4$

$y = t+1$

$z = t$

Q (10, −1, −2)

So if x, y & z all give same value of t ⇒ lies on line L

$x=10$: $x = -3t+4$

$10 = -3t+4$

$3t = -6$

$t = -2$

$y=-1$: $y = t+1$

$-1 = t+1$

$t = -2$

$z=-2$: $z = t$

$z = -2$

• 1

As $t=-2$ satisfies all ⇒ Q (10, −1, −2) must lie on the line L when $t=-2$.

c) π_1 normal, $n = (1, 1, 2)$

π_2 normal, $m = (1, -1, 4)$

$$\cos\theta = \frac{n \cdot m}{|n||m|} = \frac{\begin{pmatrix}1\\1\\2\end{pmatrix} \cdot \begin{pmatrix}1\\-1\\4\end{pmatrix}}{\sqrt{1+1+4}\,\sqrt{1+1+16}} \bullet = \frac{1-1+8}{\sqrt{6}\times\sqrt{18}} = \frac{8}{\sqrt{108}} \bullet$$

$$\therefore \cos\theta = \frac{8}{\sqrt{108}} = 0.7698\ldots$$

3

$\theta = \cos^{-1}(0.7698)$

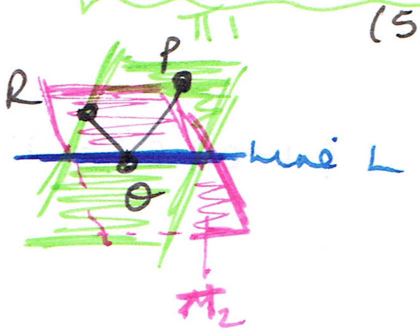
$\therefore \theta = 39.7^\circ$ •

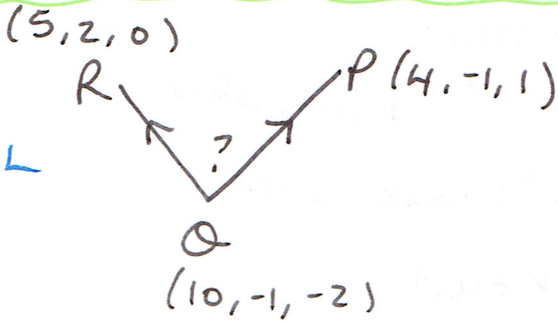
(* Must be ACUTE)

Q6 d) $\Pi_1 : x+y+2z=5 \rightarrow P(4,-1,1)$

$\Pi_2 : x-y+4z=3 \rightarrow R\ (5,2,0)$

Line intersecting Planes, $Q\ (10,-1,-2)$





$$\vec{QP} = \underline{p} - \underline{q} = \begin{pmatrix}4\\-1\\1\end{pmatrix} - \begin{pmatrix}10\\-1\\-2\end{pmatrix} = \begin{pmatrix}-6\\0\\3\end{pmatrix}$$

$$\vec{QR} = \underline{r} - \underline{q} = \begin{pmatrix}5\\2\\0\end{pmatrix} - \begin{pmatrix}10\\-1\\-2\end{pmatrix} = \begin{pmatrix}-5\\3\\2\end{pmatrix}$$

3

$$\therefore \cos(P\hat{Q}R) = \frac{\begin{pmatrix}-6\\0\\3\end{pmatrix} \cdot \begin{pmatrix}-5\\3\\2\end{pmatrix}}{\sqrt{36+0+9}\ \sqrt{25+9+4}} = \frac{30+0+6}{\sqrt{45} \times \sqrt{38}} = \frac{36}{\sqrt{1710}}$$

$$\therefore \cos(P\hat{Q}R) = 0.8706..$$

$$P\hat{Q}R = \cos^{-1}(0.8706)$$

$$\therefore P\hat{Q}R = 29.5^\circ$$

So Difference between angle of line with 2 planes $\Rightarrow 39.7 - 29.5 = 10.2^\circ$

Q7. $4\frac{d^2y}{dx^2} - 12\frac{dy}{dx} + 9y = 7\sin x + 17\cos x$

Aux Eqn: $4m^2 - 12m + 9 = 0$

$(2m-3)^2 = 0$

$\therefore \quad m = 3/2$ (twice)

Complementary Function, $y_c = Ae^{3/2x} + Bxe^{3/2x}$

Particular Integral Let $y_p = p\sin x + q\cos x$

$dy_p/dx = p\cos x - q\sin x$

$d^2y_p/dx^2 = -p\sin x - q\cos x$

$$\boxed{4\frac{d^2y}{dx^2} - 12\frac{dy}{dx} + 9y = 7\sin x + 17\cos x}$$

$4(-p\sin x - q\cos x) - 12(p\cos x - q\sin x) + 9(p\sin x + q\cos x) = 7\sin x + 17\cos x$

$(-4p + 12q + 9p)\sin x + (-4q - 12p + 9q)\cos x = 7\sin x + 17\cos x$

$\therefore \quad (5p + 12q)\sin x = 7\sin x \qquad (-12p + 5q)\cos x = 17\cos x$

$5p + 12q = 7 \quad -① \qquad -12p + 5q = 17 \quad -②$

$5p + 12q = 7 \quad -①$

$-12p + 5q = 17 \quad -②$

① ×5 $\quad 25p + 60q = 35$

② ×12 $\quad -144p + 60q = 204$

$169p = -169$

$\therefore \quad p = -1$

Subst $p = -1$ into ①

$-5 + 12q = 7$

$12q = 12$

$\therefore \quad q = 1$

$\therefore \quad y_p = -\sin x + \cos x$

$$\boxed{\begin{aligned} y_c &= Ae^{3/2x} + Bxe^{3/2x} \\ \text{\& } y_p &= \cos x - \sin x \end{aligned}}$$

7

$\therefore$ General Solution $y = y_c + y_p$

$$\boxed{y = Ae^{3/2x} + Bxe^{3/2x} + \cos x - \sin x}$$

$y = 3$ & $x = 0$:

$$3 = Ae^0 + 0 + \cos 0 - \sin 0$$

$$3 = A + 1$$

$$\Rightarrow \underline{\underline{A = 2}}$$

$\boxed{Bxe^{3/2x}}$

$u = Bx \quad v = e^{3/2x}$

$u' = B \quad v' = \frac{3}{2}e^{3/2x}$

$$\frac{dy}{dx} = \frac{3}{2}Ae^{3/2x} + \left(Be^{3/2x} + \frac{3}{2}Bxe^{3/2x}\right) - \sin x - \cos x$$

$\frac{dy}{dx} = 5/2$ & $x = 0$

3

$$\frac{5}{2} = \frac{3}{2}(2)e^0 + Be^0 + 0 - \sin 0 - \cos 0$$

$$\frac{5}{2} = 3 + B - 1$$

$$\frac{5}{2} = 2 + B \Rightarrow \underline{\underline{B = 1/2}}$$

$$\therefore \quad y = 2e^{3/2x} + \frac{1}{2}xe^{3/2x} + \cos x - \sin x$$

$\boxed{45}$