AH 2010/11 Units 1/2 Prelim

Q1. $f(x) = \ln|x| \, \text{Cos}^2(2x)$

$= \ln|x| \left(\text{Cos}(2x)\right)^2$

$u = \ln|x| \quad v = [\cos(2x)]^2$

$u' = \frac{1}{x} \quad v' = 2[\cos(2x)]^1 \cdot -2\sin(2x)$

$= -4\cos 2x \sin 2x$

$f'(x) = u'v + uv'$

$= \frac{1}{x}\left(\cos(2x)\right)^2 + \ln|x| \times -4\cos(2x)\sin(2x)$

$= \underline{\text{Cos}(2x)\left[\frac{\cos(2x)}{x} - 4\ln|x|\sin(2x)\right]}$ ④

Fine to stop here but may continue to find denominator common to both:-

$$\underline{\frac{\cos(2x)}{x}\left[\text{Cos}(2x) - 4x\ln|x|\sin(2x)\right]}$$

1- Method

2- Diff $\ln|x|$

3- Repeated use of Chain Rule

4- Correct 2 terms.

(4 marks)

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Q2.

$$y = \frac{\sin^{-1}(\sqrt{x})}{\sqrt{1-x}} = \frac{\sin^{-1}(x^{1/2})}{(1-x)^{1/2}}$$

$$\boxed{\frac{dy}{dx} = \frac{u'v-uv'}{v^2}}$$

$$\boxed{\begin{aligned} u &= \sin^{-1}(x^{1/2}) \\ u' &= \frac{1}{\sqrt{1-(\sqrt{x})^2}} \cdot \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}\sqrt{1-x}} \\ v &= (1-x)^{1/2} \\ v' &= \frac{1}{2}(1-x)^{-1/2} \cdot -1 \\ &= \frac{-1}{2\sqrt{1-x}} \end{aligned}}$$

$$= \frac{\left(\frac{1}{2\sqrt{x}\sqrt{1-x}} \cdot \sqrt{1-x} - \sin^{-1}(\sqrt{x}) \cdot \frac{-1}{2\sqrt{1-x}}\right)}{(\sqrt{1-x})^2}$$

$$= \frac{\left(\frac{1}{2\sqrt{x}\sqrt{1-x}} \times \sqrt{1-x} + \frac{\sin^{-1}(\sqrt{x})}{2\sqrt{1-x}}\right)}{(1-x)}$$

$$= \frac{\left(\frac{\sqrt{1-x}}{2\sqrt{x}\sqrt{1-x}} + \frac{\sqrt{x}}{\sqrt{x}} \times \frac{\sin^{-1}(\sqrt{x})}{2\sqrt{1-x}}\right)}{(1-x)}$$

$$= \frac{\left(\sqrt{1-x} + \sqrt{x}\sin^{-1}(\sqrt{x})\right)}{2\sqrt{x}\sqrt{1-x}\,(1-x)^{2/2}}$$

④

$$= \frac{\sqrt{1-x} + \sqrt{x}\sin^{-1}(\sqrt{x})}{2\sqrt{x(1-x)^3}}$$

As Required

Q3.

$$\begin{array}{l} x - 2y + 2z = 1 \\ 2x + 2y - z = 0 \\ 3x - 7y + 11z = 5 \end{array} \rightarrow \begin{array}{c} \left(\begin{array}{ccc|c} 1 & -2 & 2 & 1 \\ 2 & 2 & -1 & 0 \\ 3 & -7 & 11 & 5 \end{array}\right) \end{array} \begin{array}{l} r_1 \\ r_2 \\ r_3 \end{array}$$

$$\begin{array}{l} 6r_1 \\ 3r_2 \\ 2r_3 \end{array} \left(\begin{array}{ccc|c} 6 & -12 & 12 & 6 \\ 6 & 6 & -3 & 0 \\ 6 & -14 & 22 & 10 \end{array}\right) \begin{array}{l} r_1 \\ r_2 \\ r_3 \end{array} \rightarrow \begin{array}{l} \\ r_2 - r_1 \\ r_3 - r_1 \end{array} \left(\begin{array}{ccc|c} 6 & -12 & 12 & 6 \\ 0 & 18 & -15 & -6 \\ 0 & -2 & 10 & 4 \end{array}\right) \begin{array}{l} r_1 \\ r_2 \\ r_3 \end{array}$$

$$\begin{array}{l} \tfrac{1}{6}r_1 \\ r_2 \\ 9r_3 \end{array} \left(\begin{array}{ccc|c} 1 & -2 & 2 & 1 \\ 0 & 18 & -15 & -6 \\ 0 & -18 & 90 & 36 \end{array}\right) \begin{array}{l} r_1 \\ r_2 \\ r_3 \end{array} \rightarrow \begin{array}{l} \\ \\ r_2 + r_3 \end{array} \left(\begin{array}{ccc|c} 1 & -2 & 2 & 1 \\ 0 & 18 & -15 & -6 \\ 0 & 0 & 75 & 30 \end{array}\right) \begin{array}{l} r_1 \\ r_2 \\ r_3 \end{array}$$

$\Rightarrow$ $75z = 30$

$z = \frac{30}{75}$

$\therefore \underline{z = \frac{2}{5}}$

$18y - 15z = -6$

$6y - 5z = -2$

$6y - 5\left(\frac{2}{5}\right) = -2$

$6y - 2 = -2$

$6y = 0$

$\therefore \underline{y = 0}$

$x - 2y + 2z = 1$

$x - 0 + 2\left(\frac{2}{5}\right) = 1$

$x + 4/5 = 1$

$\underline{x = 1/5}$

⑤

$\therefore \quad x = 1/5 \; ; \; y = 0 \; ; \; z = \frac{2}{5}$

So intersect at $(1/5, 0, 2/5)$

Q4 (a) $$\frac{3x^2+26x+48}{x(x+4)^2} = \frac{A}{x} + \frac{B}{(x+4)} + \frac{C}{(x+4)^2}$$

$$\therefore \quad 3x^2+26x+48 = A(x+4)^2 + Bx(x+4) + Cx$$

Let $x = 0$: $48 = 16A$

$$\Rightarrow \underline{\underline{A = 3}}$$

Let $x = -4$: $3(-4)^2+26(-4)+48 = C(-4)$

$$48 - 104 + 48 = -4C$$

$$-8 = -4C$$

$$\Rightarrow \quad \underline{\underline{C = 2}}$$

Let $x = 1$: $3 + 26 + 48 = A(5)^2 + B(5) + C$

$$77 = 25(3) + 5B + 2$$

$$77 = 75 + 5B + 2$$

$$5B = 0$$

$$\Rightarrow \underline{\underline{B = 0}}$$

③

$$\boxed{\therefore \quad \frac{3x^2+26x+48}{x(x+4)^2} = \underline{\underline{\frac{3}{x} + \frac{2}{(x+4)^2}}}}$$

(b) $$\int\left(\frac{3}{x} + 2(x+4)^{-2}\right)dx = 3\ln|x| + \frac{2(x+4)^{-1}}{-1} + c$$

$$= \underline{\underline{\ln|x^3| - \frac{2}{(x+4)} + c}}$$

③

Q5. If $x = 3+i$ is a root $R=0$ in Synthetic Division

$$x^3 - 8x^2 + 22x - 20 = 0$$

$$\begin{array}{c|cccc} & 1 & -8 & 22 & -20 \\ 3+i & \downarrow & 3+i & -16-2i & 20 \\ \hline & 1 & -5+i & 6-2i & \underline{\underline{0}} \end{array}$$

$$(3+i)(-5+i) = -15 + 3i - 5i + i^2 = -16-2i$$

$$(3+i)(6-2i) = 18 - 6i + 6i - 2i^2 = 20$$ ②

- As Remainder, $R=0 \Rightarrow x = 3+i$ is a solution.
- Thus the conjugate pair $\underline{\bar{x} = 3-i}$ is also a solution ①.

OR

If $x = 3+i$

$$x^2 = (3+i)(3+i) = 9 + 3i + 3i + i^2 = 9 + 6i - 1 = \underline{\underline{8+6i}}$$

$$x^3 = (3+i)(8+6i) = 24 + 18i + 8i + 6i^2 = 24 + 26i - 6 = \underline{\underline{18+26i}}$$

$$x^3 - 8x^2 + 22x - 20$$

$$= (18+26i) - 8(8+6i) + 22(3+i) - 20$$

$$= 18 + 26i - 64 - 48i + 66 + 22i - 20$$

$$= (18 - 64 + 66 - 20) + i(26 - 48 + 22)$$

$$= (84 - 84) + i(48 - 48)$$

$$= \underline{\underline{0}}$$

(Alternative ②)

As Substituting $x = 3+i$ fits perfectly + no Remainder

$\Rightarrow$ $x = 3+i$ is a solution.

$x = 3+i$ & $x = 3-i \Rightarrow 2$ factors $(x-3-i)$ & $(x-3+i)$

Multiplying $(x-3-i)(x-3+i) = \underline{x^2 - 6x + 10}$

	x	-3	$-i$
x	x^2	$-3x$	$\cancel{-ix}$
-3	$-3x$	$+9$	$\cancel{+3i}$
$+i$	$\cancel{+ix}$	$\cancel{-3i}$	$-i^2(=+1)$

Easier to multiply with a grid to obtain quadratic made of 2 factors.

To find final factor need long division:

$$\begin{array}{r|l} & x \quad -2 \\ \hline x^2-6x+10 & x^3 - 8x^2 + 22x - 20 \\ & x^3 - 6x^2 + 10x \\ \hline & \qquad -2x^2 + 12x - 20 \\ & \qquad -2x^2 + 12x - 20 \\ \hline & \end{array}$$

$\Rightarrow$ final factor is $\underline{(x-2)}$

So $x^3 - 8x^2 + 22x - 20 = (x-2)(x-3-i)(x-3+i)$
$= (x-2)(x^2-6x+10)$
(where $a=2$, $b=-6$ & $c=10$) ②

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and 3 roots are

$\underline{x = 2;\ x = 3+i\ \&\ x = 3-i}$ ①

Q6. AH 2010/11 Units 1/2 Prelim

$$\int_{-1}^{0} \frac{dx}{5-\sqrt{3x+4}} = \int_{-4}^{-3} \frac{2/3\,(\theta+5)\,d\theta}{5-\sqrt{(\theta+5)^2}}$$

If $3x+4 = (\theta+5)^2$

$$3x = (\theta+5)^2-4$$

$$x = \frac{(\theta+5)^2-4}{3}$$

$$\frac{dx}{d\theta} = \frac{2(\theta+5)'}{3}$$

$$\boxed{dx = \frac{2}{3}(\theta+5)\,d\theta}$$

If $x=0$: $0+4 = (\theta+5)^2$

$$(\theta+5)^2 = 4$$

$$\theta+5 = 2$$

$$\underline{\underline{\theta = -3}}$$

If $x=-1$: $-3+4 = (\theta+5)^2$

$$1 = (\theta+5)^2$$

$$\theta+5 = 1$$

$$\underline{\underline{\theta = -4}}$$

$$= \frac{2}{3}\int_{-4}^{-3} \frac{(\theta+5)\,d\theta}{5-(\theta+5)}$$

$$= \frac{2}{3}\int_{-4}^{-3} \frac{\theta+5}{(5-\theta-5)}\,d\theta$$

$$= \frac{2}{3}\int_{-4}^{-3} \left(\frac{\theta+5}{-\theta}\right) d\theta$$

$$= \frac{2}{3}\int_{-4}^{-3} \left(-1-\frac{5}{\theta}\right) d\theta$$

$$= \frac{2}{3}\left[-\theta-5\ln\theta\right]_{-4}^{-3}$$

(Could pull out negative but not nec.)

$$= \frac{2}{3}\left[\left(-(-3)-5\ln|-3|\right)-\left(-(-4)-5\ln|-4|\right)\right]$$

$$= \frac{2}{3}\left(3-5\ln|3|-4+5\ln|4|\right)$$

$$= \frac{2}{3}\left(5\left(\ln|4|-\ln|3|\right)-1\right)$$

$$= \frac{10}{3}\ln\left|\frac{4}{3}\right| - \frac{2}{3}$$

$$= \frac{10}{3}\ln\left|\frac{4}{3}\right| - \frac{2}{3}\times\frac{5}{5}$$

[Have to get $\frac{10}{3}$ as common factor for final answer]

$$= \frac{10}{3}\ln\left|\frac{4}{3}\right| - \frac{10}{3}\times\frac{1}{5}$$

$$= \underline{\underline{\frac{10}{3}\left(\ln\left|\frac{4}{3}\right| - \frac{1}{5}\right)}}$$

As required.

6

AH 2010/11 Unit 1/2 Prelim

Q7 (a)

$x = t^3 - 1/t = t^3 - t^{-1}$

$dx/dt = 3t^2 + t^{-2}$

$= 3t^2 + \frac{1}{t^2}$

$y = \frac{-4}{t^2} + t = -4t^{-2} + t$

$dy/dt = 8t^{-3} + 1$

$= \frac{8}{t^3} + 1$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{\frac{8}{t^3}+1}{3t^2+\frac{1}{t^2}} = \frac{\frac{8+t^3}{t^3}}{\frac{3t^4+1}{t^2}} = \frac{8+t^3}{t^3} \times \frac{t^2}{3t^4+1}$$

$$= \frac{8+t^3}{t(3t^4+1)}$$

Stat Pt $\frac{dy}{dx} = 0 \Rightarrow \frac{8+t^3}{t(3t^4+1)} = 0$

$8+t^3 = 0$

$t^3 = -8$

$\therefore t = -2$

④

Co-ordinate of stat Pt at $t=-2$:

$x = t^3 - \frac{1}{t} = (-2)^3 - \frac{1}{(-2)} = -8 + 1/2 = -7\frac{1}{2}$ or $\frac{-15}{2}$

$y = \frac{-4}{t^2} + t = \frac{-4}{(-2)^2} + (-2) = \frac{-4}{4} - 2 = -3$

$\left(\frac{-15}{2}, -3\right)$

Q7(b)

$$\frac{dx}{dt} = 3t^2 + \frac{1}{t^2} = \frac{3t^4+1}{t^2}$$

$$\frac{dy}{dx} = \frac{8+t^3}{t(3t^4+1)} = \frac{8+t^3}{3t^5+t}$$

$$\boxed{\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}}}$$

Firstly need ↓

$$\frac{d}{dt}\left(\frac{dy}{dx}\right) = \left(\frac{u'v - uv'}{v^2}\right)$$

$$\boxed{\begin{array}{ll} u = 8+t^3 & v = 3t^5+t \\ u' = 3t^2 & v' = 15t^4+1 \end{array}}$$

$$= \frac{3t^2(3t^5+t) - (8+t^3)(15t^4+1)}{(3t^5+t)^2}$$

$$= \frac{9t^7 + 3t^3 - 120t^4 - 8 - 15t^7 - t^3}{(t(3t^4+1))^2}$$

$$= \frac{-6t^7 - 120t^4 + 2t^3 - 8}{t^2(3t^4+1)^2}$$

$$= \frac{-2(3t^7+60t^4-t^3+4)}{t^2(3t^4+1)^2}$$

④

$$\frac{d^2y}{dx^2} = \frac{-2(3t^7+60t^4-t^3+4)}{t^2(3t^4+1)^2} \div \frac{(3t^4+1)}{t^2}$$

$$= \frac{-2(3t^7+60t^4-t^3+4)}{t^2(3t^4+1)^2} \times \frac{t^2}{(3t^4+1)}$$

$$= \frac{-2(3t^7+60t^4-t^3+4)}{(3t^4+1)^3}$$

$\underline{\text{At } t=-2}$ $= \frac{-2(3(-2)^7+60(-2)^4-(-2)^3+4)}{(3(-2)^4+1)^3} = \underline{-1176}$ $<0 \Rightarrow \cap$ Max TPt $\left(\frac{-15}{2}, -3\right)$

Q8.

$\underset{\sim\sim\sim}{u_1 = a = 7 \quad S_4 = 52}$: $\quad S_n = \frac{n}{2}\left[2a+(n-1)d\right]$.

$52 = \frac{4}{2}\left[2\times7+(4-1)d\right]$ ●

$52 = 2\left[14+3d\right]$

$52 = 28+6d$

$6d = 24$

$\therefore \underline{\underline{d = 4}}$ ●

$S_{20} = \frac{20}{2}\left[2\times7+(20-1)\times4\right]$ ●

$= 10\left[14+19\times4\right]$

$= 10(14+76)$

$= 10\times90$

$\therefore S_{20} = \underline{\underline{900}}$ ●

④

AH 2010/11 Units 1&2 Prelim

Q9. $\int 5\ln|x^2+9|\,dx$

$$\boxed{\int u'v = uv - \int uv'}$$

$u = 5x \qquad v = \ln|x^2+9|$

$u' = 5 \qquad v' = \frac{1}{x^2+9}\cdot 2x$ ●

$= 5x\ln|x^2+9| - \int \frac{10x^2}{x^2+9}\,dx$ ●

$= 5x\ln|x^2+9| - 10\int \frac{x^2\,dx}{x^2+9}$

$= 5x\ln|x^2+9| - 10\int \frac{(x^2+9-9)}{x^2+9}\,dx$

OR

$$\begin{array}{r|l} & 1 \\ \hline x^2+9 & x^2 \\ & x^2+9 \\ \hline & -9 \end{array}$$

$= \left(1 - \frac{9}{x^2+9}\right)$

$= 5x\ln|x^2+9| - 10\int\left(\frac{x^2+9}{x^2+9} - \frac{9}{x^2+9}\right)dx$

$= 5x\ln|x^2+9| - 10\int\left(1 - \frac{9}{(x^2+3^2)}\right)dx$ ●

$= 5x\ln|x^2+9| - 10\int dx + 90\int\frac{dx}{x^2+3^2}$

$= 5x\ln|x^2+9| - 10x + 90\left(\frac{\tan^{-1}(x/3)}{(3)}\right) + c$

Chain Rule

$= \underline{\underline{5x\ln|x^2+9| - 10x + 30\tan^{-1}(1/3) + c}}$

● ● ●

⑥

Q10.

$$\frac{dy}{dt} = \frac{\sec y}{4e^{3t}}$$

$$\frac{dy}{\sec y} = \frac{dt}{4e^{3t}}$$

- Change sec y to $\left(\frac{1}{\cos y}\right)$
- Need exp on numerator
- Separate variables then integrate

$$\frac{dy}{\left(\frac{1}{\cos y}\right)} = \frac{1}{4}e^{-3t}\,dt$$

$$\therefore \int \cos y\, dy = \int \frac{1}{4}e^{-3t}\,dt$$

$$\sin y = \frac{1}{4}\cdot\frac{e^{-3t}}{-3} + c$$

$$\sin y = \frac{-1}{12e^{3t}} + c$$

(Can rearrange further, but easier to find c if subst. in here)

$$\left.\begin{array}{l}\underline{\text{At } y=\frac{\pi}{6}} \\ \underline{+ \quad t=0}\end{array}\right\} \sin\left(\frac{\pi}{6}\right) = \frac{-1}{12e^{0}} + c$$

$$\frac{1}{2} = \frac{-1}{12} + c$$

$$\therefore \quad c = 1/2 + \frac{1}{12} = \frac{6}{12} + \frac{1}{12} = \underline{\underline{\frac{7}{12}}}$$

$$\therefore \quad \sin y = \frac{-1}{12e^{3t}} + \frac{7}{12}$$

$$\text{So} \quad \underline{\underline{y = \sin^{-1}\left(\frac{7 - e^{-3t}}{12}\right)}}$$

or some alternative variation,

$$y = \sin^{-1}\left(\frac{7}{12} - \frac{1}{12e^{3t}}\right)$$

⑤

Q11.(a) $k=1$ •

As $x=1$ is Vertical Asymptote $\Rightarrow f(x)=\frac{x^2+8}{x-1}$

①

(b) $f(x)=\frac{x^2+8}{x-1} = x+1+\frac{9}{(x-1)}$ •

$$\begin{array}{r|l} & x+1 \\ x-1 & x^2+0x+8 \\ & x^2-x \\ \hline & x+8 \\ & x-1 \\ \hline & \underline{\underline{9}} \end{array}$$

•

$\therefore$ $y=x+1$ is Non-Vertical Asymptote •

③

(c) $x=0$: $y=\frac{0^2+8}{0-1} = \underline{\underline{-8}} \Rightarrow P(0,-8)$ •

①

(d) $f(x)=x+1+9(x-1)^{-1}$

$f'(x) = 1-9(x-1)^{-2} = 1-\frac{9}{(x-1)^2}$ •

Stat Pts at $f'(x)=0$: $1-\frac{9}{(x-1)^2}=0$

$1=\frac{9}{(x-1)^2}$

②

$(x-1)^2=9$

$x-1=\pm 3$

$x=1-3;\ 1+3$

$\therefore$ $x=-2;\ x=4$ •

• $f'(x)=0$ & solves for x correctly.

Q11.

(d) (ctd/...)

$$\boxed{\begin{aligned} f(x) &= x+1+9(x-1)^{-1} = x+1+\frac{9}{(x-1)} \\ f'(x) &= 1-9(x-1)^{-2} = 1-\frac{9}{(x-1)^2} \\ f''(x) &= 18(x-1)^{-3} = \frac{18}{(x-1)^3} \end{aligned}}$$

Stat Pts at $x=-2$ & $x=4$

y co-ords:

$f(-2) = -2+1+\frac{9}{(-2-1)} = -1+\frac{9}{-3} = -1-3 = \underline{\underline{-4}} \Rightarrow (-2,-4)$

$f(4) = 4+1+\frac{9}{(4-1)} = 5+\frac{9}{3} = 5+3 = \underline{\underline{8}} \Rightarrow (4,8)$ ①

Nature $f''(x) = \frac{18}{(x-1)^3}$

②

At $x=-2$: $f''(-2) = \frac{18}{(-2-1)^3} = \frac{18}{-27} < 0 \Rightarrow \cap$ Max $(-2,-4)$

At $x=4$: $f''(4) = \frac{18}{(4-1)^3} = \frac{18}{27} > 0 \Rightarrow \cup$ Mini $(4,8)$

Given Stat Pts $\rightarrow |f(x)| \rightarrow -|f(x)| \rightarrow -|f(x)|+2 = 2-|f(x)|$

$(-2,-4) \rightarrow (-2,4) \rightarrow (-2,-4) \rightarrow (-2,-2)$

$(4,8) \rightarrow (4,8) \rightarrow (4,-8) \rightarrow (4,-6)$ ②