2009/10 AH Mini Prelim

Q1.
a)

$$AB = \begin{pmatrix} p & 1 \\ -2 & q \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 0 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 2p+0 & p-1 \\ -4+0 & -2-q \end{pmatrix}$$

$$\therefore AB = \begin{pmatrix} 2p & p-1 \\ -4 & -2-q \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ -4 & 2 \end{pmatrix}$$

$\therefore 2p = 2$ & $-2-q = 2$

$\underline{\underline{p=1}}$ $\quad -q = 4$

$\underline{\underline{q=-4}}$

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b) Given $p=1$ & $q=-4$

$$BA = \begin{pmatrix} 2 & 1 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ -2 & -4 \end{pmatrix} = \begin{pmatrix} 2-2 & 2-4 \\ 0+2 & 0+4 \end{pmatrix} = \begin{pmatrix} 0 & -2 \\ 2 & 4 \end{pmatrix}$$

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c) $C(AB) = BA$

(Alternative strategy in Mark Scheme)

$$\boxed{\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 2 & 0 \\ -4 & 2 \end{pmatrix} = \begin{pmatrix} 0 & -2 \\ 2 & 4 \end{pmatrix}}$$

$2a - 4b = 0$

$2a = 4b$

$\underline{\underline{a = 2b}}$

& $2b = -2$

$\underline{\underline{b = -1}}$

$\therefore \underline{\underline{a = -2}}$

$2c - 4d = 2$

$\underline{c - 2d = 1}$

& $2d = 4$

$\therefore \underline{\underline{d = 2}}$

& $c - 4 = 1$

$\therefore \underline{\underline{c = 5}}$

$\Rightarrow$ If $C = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$$\therefore C = \begin{pmatrix} -2 & -1 \\ 5 & 2 \end{pmatrix}$$

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Q2. $x_{n+1} = \dfrac{14 - 5x_n}{x_n}$

$$\boxed{x = \frac{14 - 5x}{x}}$$

Last few questions set $x_n, x_{n+1} \to \lambda$ But can simply call x if prefer

$x^2 = 14 - 5x$

$x^2 + 5x - 14 = 0$

$(x-2)(x+7) = 0$

2 fixed Points at $x = 2$ & $x = -7$

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Q3. $L_1 : \dfrac{x-1}{1} = \dfrac{y}{-3} = \dfrac{z+3}{4} \; (=\lambda) \Rightarrow$ $L_1 : x = 1+\lambda$, $y = -3\lambda$, $z = -3+4\lambda$

$L_2 : \dfrac{x-4}{1} = \dfrac{y+5}{-1} = \dfrac{z-5}{2} \; (=\mu) \Rightarrow$ $L_2 : x = 4+\mu$, $y = -5-\mu$, $z = 5+2\mu$

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Let $y=y$: $-3\lambda = -5-\mu$.

$\therefore \mu = 3\lambda - 5$

Let $x=x$

$1+\lambda = 4+\mu$

$1+\lambda = 4+(3\lambda-5)$

$1+\lambda = 3\lambda - 1$

$-2\lambda = -2$

$\therefore \lambda = 1$

& $\mu = 3-5 = -2$

$\lambda = 1$

L_1: $x = 1+1 = 2$

$y = -3(1) = -3$

$z = -3+4 = 1$

$\mu = -2$

L_2: $x = 4-2 = 2$

$y = -5+2 = -3$

$z = 5-4 = 1$

$\Rightarrow$ L_1 & L_2 intersect at $(2,-3,1)$

Q3 (ii)

$L_1: \frac{x-1}{1} = \frac{y}{-3} = \frac{z+3}{4} \Rightarrow \underline{d}_1 = \underline{i} - 3\underline{j} + 4\underline{k}$

$L_2: \frac{x-4}{1} = \frac{y+5}{-1} = \frac{z-5}{2} \Rightarrow \underline{d}_2 = \underline{i} - \underline{j} + 2\underline{k}$ •

Angle $\cos\theta = \frac{\underline{d}_1 \cdot \underline{d}_2}{|\underline{d}_1||\underline{d}_2|} = \frac{\begin{pmatrix}1\\-3\\4\end{pmatrix} \cdot \begin{pmatrix}1\\-1\\2\end{pmatrix}}{\sqrt{1+9+16}\sqrt{1+1+4}} = \frac{1+3+8}{\sqrt{26}\sqrt{6}}$ •

$\therefore \cos\theta = \frac{12}{\sqrt{156}}$

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$\theta = \cos^{-1}(0.9608)$

$\theta = 16.1^\circ$ •

(iii) $\text{Normal} = \underline{d}_1 \times \underline{d}_2 = \begin{pmatrix} \underline{i} & \underline{j} & \underline{k} \\ 1 & -3 & 4 \\ 1 & -1 & 2 \end{pmatrix}$ •

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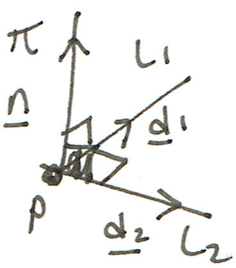
$\underline{n} = \underline{i}\begin{vmatrix}-3 & 4\\-1 & 2\end{vmatrix} - \underline{j}\begin{vmatrix}1 & 4\\1 & 2\end{vmatrix} + \underline{k}\begin{vmatrix}1 & -3\\1 & -1\end{vmatrix}$

$= (6-(-4))\underline{i} - (2-4)\underline{j} + (-1-(-3))\underline{k}$

$\therefore \underline{n} = -2\underline{i} + 2\underline{j} + 2\underline{k}$ • → (can reduce to $\underline{n} = \underline{i} - \underline{j} - \underline{k}$ if wish at this point)



$\boxed{\underline{r} \cdot \underline{n} = \underline{a} \cdot \underline{n}}$

$\begin{pmatrix}x\\y\\z\end{pmatrix} \cdot \begin{pmatrix}-2\\2\\2\end{pmatrix} = \begin{pmatrix}2\\-3\\1\end{pmatrix} \cdot \begin{pmatrix}-2\\2\\2\end{pmatrix}$

$-2x + 2y + 2z = -4 - 6 + 2 = -8$

$-x + y + z = -4$ (Both fine)

OR •

$\therefore x - y - z = 4$ is equation of plane

Q4. $\boxed{\sum_{r=1}^{n} \frac{1}{r(r+1)} = \frac{n}{n+1}}$

Let $n=1$: LHS $= \frac{1}{1(1+1)} = \frac{1}{2}$ RHS $= \frac{1}{(1+1)} = \frac{1}{2}$ LHS = RHS ✓

True for $n=1$

Assume true for $n=k$: $\sum_{r=1}^{n=k} \frac{1}{r(r+1)} = \frac{k}{k+1}$

Consider $n=k+1$:

$$\sum_{r=1}^{n=k+1} \frac{1}{r(r+1)} = \sum_{r=1}^{n=k} \frac{1}{r(r+1)} \boxed{+ \frac{1}{(k+1)((k+1)+1)}} = \frac{k}{k+1} \boxed{+ \frac{1}{(k+1)((k+1)+1)}}$$

$$= \frac{k}{k+1} + \frac{1}{(k+1)(k+2)}$$

$$= \frac{k(k+2)}{(k+1)(k+2)} + \frac{1}{(k+1)(k+2)}$$

$$= \frac{k^2+2k+1}{(k+1)(k+2)}$$

$$= \frac{(k+1)^2}{(k+1)(k+2)}$$

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$$= \frac{(k+1)}{(k+2)}$$

$$= \frac{(k+1)}{((k+1)+1)}$$ As required

As true for $n=1$, also assumed true for $n=k$ and by Proof of Mathematical Induction also true for $n=k+1$. Conjecture must be true $\forall n \in N$.

Q5. $432_5 \rightarrow$

5^2	5^1	5^0
4	3	2

$5^2 \times 4 + 5^1 \times 3 + 5^0 \times 2$

$= 25 \times 4 + 5 \times 3 + 1 \times 2$

$= \underline{\underline{117}}$

$117 = 16 \cdot 7 + 5$

$16 = 2 \cdot 7 + 2$

$2 = 0 \cdot 7 + 2$

(3)

$\therefore \; 432_5 = 225_7$

Q6 $\boxed{f(x) \simeq f(0) + f'(0)\frac{x}{1!} + f''(0)\frac{x^2}{2!} + f'''(0)\frac{x^3}{3!} + \ldots + f^n(0)\frac{x^n}{n!}}$

(*) $f(x) = \ln\left(1 - \frac{x}{2}\right) = \ln\left(\frac{2-x}{2}\right) = \ln(2-x) - \ln 2$

$f'(x) = \frac{1}{(2-x)} \cdot -1 = \frac{-1}{(2-x)} = -(2-x)^{-1} = \frac{-1}{(2-x)}$

$f''(x) = (2-x)^{-2} \cdot -1 = -(2-x)^{-2} = \frac{-1}{(2-x)^2}$

$f'''(x) = 2(2-x)^{-3} \cdot -1 = -2(2-x)^{-3} = \frac{-2}{(2-x)^3}$

$f^{iv}(x) = 6(2-x)^{-4} \cdot -1 = -6(2-x)^{-4} = \frac{-6}{(2-x)^4}$

$f(0) = \underline{\underline{0}}$

$f'(0) = \underline{\underline{\frac{-1}{2}}}$

$f''(0) = \underline{\underline{\frac{-1}{4}}}$

$f'''(0) = \frac{-2}{8} = \underline{\underline{\frac{-1}{4}}}$

$f^{iv}(0) = \frac{-6}{16} = \underline{\underline{\frac{-3}{8}}}$

$\therefore \; \ln\left(1 - \frac{x}{2}\right) \simeq 0 + \left(\frac{-1}{2}\right)\frac{x}{1!} + \left(\frac{-1}{4}\right)\frac{x^2}{2!} + \left(\frac{-1}{4}\right)\frac{x^3}{3!} + \left(-\frac{3}{8}\right)\frac{x^4}{4!} + \ldots$

$= -\frac{x}{2} - \frac{x^2}{8} - \frac{x^3}{24} - \frac{x^4}{64}$ (first 4 terms) (5)

(*) $\boxed{\ln\left(1 - \frac{x}{2}\right) = \ln\left(\frac{2-x}{2}\right) = \ln(2-x) - \ln(2)}$ from above.

So $\ln(2-x) = \ln\left(1 - \frac{x}{2}\right) + \ln(2)$

$= \ln(2) - x/2 - x^2/8 - x^3/24 - x^4/64$

Q7. $$\boxed{\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 5y = 8 - 5x}$$

Aux Eqn $m^2 + 2m + 5 = 0$

$$\boxed{m = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}}$$

$$m = \frac{-2 \pm \sqrt{4 - 4\times1\times5}}{2\times1}$$

$$= \frac{-2 \pm \sqrt{-16}}{2}$$

$$= \frac{-2 \pm 4i}{2}$$

$$\therefore m = -1 \pm 2i$$

$(m = p \pm qi)$

Complex Solutions:

$$y_c = e^{px}(A\cos qx + B\sin qx)$$

$$y_c = e^{-x}(A\cos 2x + B\sin 2x)$$

Particular Integral $y_p = ax + b$

$dy_p/dx = a$ & $\frac{d^2y_p}{dx^2} = 0$

$$\therefore \frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 5y = 8 - 5x$$

$$0 + 2(a) + 5(ax+b) = 8 - 5x$$

$$5ax + (2a + 5b) = 8 - 5x$$

$\therefore$ $5ax = -5x$ & $2a + 5b = 8$

$a = -1$ $\quad -2 + 5b = 8$

$5b = 10$

$\therefore b = 2$

Particular Integral

$$y_p = ax + b$$

$$\Rightarrow y_p = -x + 2$$

General Solution: $y = y_c + y_p$

$$y = e^{-x}(A\cos 2x + B\sin 2x) - x + 2$$

Q7 (continued)

$$y = e^{-x}(A\cos 2x + B\sin 2x) - x + 2$$

$x=0$, $y=5$ $\rightarrow$ $5 = e^0(A\cos 0 + B\sin 0) - 0 + 2$

$5 = 1(A+0)+2$

$5 = A+2$

$\Rightarrow$ $A = 3$

$u = e^{-x}$ $\quad v = A\cos 2x + B\sin 2x$

$u' = -e^{-x}$ $\quad v' = -2A\sin 2x + 2B\cos 2x$

$$\frac{dy}{dx} = e^{-x}(-2A\sin 2x + 2B\cos 2x) - e^{-x}(A\cos 2x + B\sin 2x) - 1$$

$x=0$, $\frac{dy}{dx} = -6$ $\rightarrow$ $-6 = e^0(-2A\sin 0 + 2B\cos 0) - e^0(A\cos 0 + B\sin 0) - 1$

$-6 = 1(0+2B) - 1(A+0) - 1$

$-6 = 2B - A - 1$

$(A=3)$ $\quad -6 = 2B - 4$

$2B = -2$

$\Rightarrow$ $B = -1$

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Then Particular Solution is $y = y_c + y_p$

$$y = e^{-x}(3\cos 2x - \sin 2x) - x + 2$$

Q8.

$$3A-2B = 3\begin{pmatrix} 3 & 1 & 1 \\ 0 & 3 & \alpha \\ 1 & 0 & 1 \end{pmatrix} - 2\begin{pmatrix} 4 & 1 & 2 \\ -1 & 4 & 3 \\ 2 & 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 9 & 3 & 3 \\ 0 & 9 & 3\alpha \\ 3 & 0 & 3 \end{pmatrix} + \begin{pmatrix} -8 & -2 & -4 \\ 2 & -8 & -6 \\ -4 & -2 & 0 \end{pmatrix}$$

$$\therefore 3A-2B = \begin{pmatrix} 1 & 1 & -1 \\ 2 & 1 & (3\alpha-6) \\ -1 & -2 & 3 \end{pmatrix}$$

$$\boxed{\text{If Singular} \Rightarrow \text{Det}(3A-2B) = 0}$$

$$\begin{pmatrix} 1 & 1 & -1 \\ 2 & 1 & (3\alpha-6) \\ -1 & -2 & 3 \end{pmatrix} = 1\begin{vmatrix} 1 & (3\alpha-6) \\ -2 & 3 \end{vmatrix} - 1\begin{vmatrix} 2 & (3\alpha-6) \\ -1 & 3 \end{vmatrix} - 1\begin{vmatrix} 2 & 1 \\ -1 & -2 \end{vmatrix}$$

$$= 1\left(3 + 2(3\alpha-6)\right) - 1\left(\cancel{6} + (3\alpha - \cancel{6})\right) - 1\left(-4-(-1)\right)$$

$$= (3 + 6\alpha - 12) - 3\alpha + 3$$

$$= 3\alpha - 6$$

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$$\boxed{\text{Det}(3A-2B) = 0} \Rightarrow 3\alpha - 6 = 0$$

$$3\alpha = 6$$

$$\alpha = 2$$

TOTAL

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