AH 2009/10 Units 1/2 Prelim

$$f(x) = \cos^{-1}(\sqrt{x})\, e^{\left(\frac{4+x}{4-x}\right)}$$

$$(u) \qquad (v)$$

* Need to use Chain Rule & Quotient Rule

For $\sqrt{x}$: $\sqrt{x} = x^{1/2} \rightarrow \frac{d(\sqrt{x})}{dx} = \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}}$

For $\left(\frac{4+x}{4-x}\right)$ $\quad u = 4+x \quad v = 4-x \quad u' = 1 \quad v' = -1$

$$\frac{u'v - uv'}{v^2} = \frac{(4-x)-(-(4+x))}{(4-x)^2} = \frac{8}{(4-x)^2}$$

Now back to the question:

$$f(x) = \cos^{-1}(\sqrt{x})\, e^{\frac{4+x}{4-x}}$$

Let $u = \cos^{-1}(\sqrt{x})$

$$u' = \frac{-1}{\sqrt{1-(\sqrt{x})^2}} \cdot \frac{1}{2\sqrt{x}}$$

$$v = e^{(4+x/4-x)}$$

$$v' = \frac{8}{(4-x)^2} e^{\frac{4+x}{4-x}}$$

$$\boxed{f'(x) = u'v + uv'}$$

$$= \frac{-1}{2\sqrt{x}\sqrt{1-x}} \cdot e^{\frac{4+x}{4-x}} + \frac{8\cos^{-1}(\sqrt{x})e^{\frac{4+x}{4-x}}}{(4-x)^2}$$

$$= e^{\frac{4+x}{4-x}}\left[\frac{-1}{2\sqrt{x(1-x)}} + \frac{8\cos^{-1}(\sqrt{x})}{(4-x)^2}\right]$$

(No need to find a common Denominator as complicated enough!)

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Q2. $$\left(\frac{2}{u^3}-3u\right)^8 = \sum_{r=0}^{8}\binom{8}{r}\left(\frac{2}{u^3}\right)^{8-r}(-3u)^r$$

$$= \sum_{r=0}^{8}\underbrace{\binom{8}{r}(2)^{8-r}\times(-3)^r}_{\text{Coefficient}} \times \underbrace{(u^{-3})^{8-r}\times(u)^r}_{\text{term to be Independent}}$$

$$(u^{-3})^{8-r}(u)^r = u^0$$

$$u^{-24+3r}\times u^r = u^0$$

$$u^{-24+4r} = u^0$$

$$-24+4r=0$$

$$4r=24$$

$$r=6$$

If $r=6$: $C = \binom{8}{6}(2)^{8-6}(-3)^6$

$$= \frac{8!}{6!.2!}\times(2)^2\times(-3)^6$$

$$= \frac{8\times7\times\cancel{6!}}{\cancel{6!}\times2}\times4\times729$$

$$= 81648$$

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$\therefore$ the term independent of u has coefficient, $C = 81{,}648$

Q3 a)

$$f(x) = \frac{6x^4 + x^3 - 5x - 4}{x^3 - x}$$

$$\begin{array}{r|l} & 6x + 1 \\ \hline x^3 - x & 6x^4 + x^3 + 0x^2 - 5x - 4 \\ & 6x^4 \quad\quad - 6x^2 \\ \hline & x^3 + 6x^2 - 5x - 4 \\ & x^3 \quad\quad - x \\ \hline & 6x^2 - 4x - 4 \end{array}$$

$$= 6x + 1 + \frac{6x^2 - 4x - 4}{\underbrace{x(x^2-1)}}$$

$$\downarrow$$

$$\boxed{\frac{6x^2 - 4x - 4}{x(x^2-1)} = \frac{A}{x} + \frac{B}{(x-1)} + \frac{C}{(x+1)}}$$

$$\therefore \quad A(x-1)(x+1) + Bx(x+1) + Cx(x-1) = 6x^2 - 4x - 4$$

Let $x = 0$: $A(-1)(1) + 0 + 0 = -4$

$$-A = -4$$

$$\therefore A = 4$$

Let $x = 1$: $B(1)(1+1) = 6 - 4 - 4$

$$2B = -2$$

$$\therefore B = -1$$

Let $x = -1$: $C(-1)(-1-1) = 6(-1)^2 - 4(-1) - 4$

$$2C = 6 + 4 - 4$$

$$2C = 6$$

$$\therefore C = 3$$

$$\therefore f(x) = 6x + 1 + \frac{4}{x} - \frac{1}{(x-1)} + \frac{3}{(x+1)}$$

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Q3(b) $\int_2^3 f(x)\,dx$

$$= \int_2^3 \left(6x+1+\frac{4}{x}-\frac{1}{(x-1)}+\frac{3}{(x+1)}\right)dx$$

$$= \left[\frac{6x^2}{2}+x+4\ln|x|-\ln|x-1|+3\ln|x+1|\right]_2^3$$

$$= \left[3x^2+x+\ln|x|^4-\ln|x-1|+\ln|x+1|^3\right]_2^3$$

$$= \left[3x^2+x+\ln\left|\frac{x^4(x+1)^3}{(x-1)}\right|\right]_2^3$$

$$= \left(3(3)^2+3+\ln\left|\frac{(3)^4(4)^3}{(2)}\right|\right)-\left(3(2)^2+2+\ln\left|\frac{(2)^4(3)^3}{1}\right|\right)$$

$$= \left(27+3+\ln\left|\frac{81\times 64}{2}\right|\right)-\left(12+2+\ln|16\times 27|\right)$$

$$= 30+\ln|81\times 32|-14-\ln|16\times 27|$$

$$= 16+\ln\left|\frac{\cancel{81}^{3}\times\cancel{32}^{2}}{\cancel{27}\times\cancel{16}}\right|$$

$$= \underline{\underline{16+\ln|6|}} \quad \text{as required.}$$

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Q4. (a) $z = \dfrac{4-2i}{3+i} - (1-2i)(3+i)$

$= \dfrac{(4-2i)}{(3+i)} \times \dfrac{(3-i)}{(3-i)} - [3 + i - 6i - 2i^2]$

$= \dfrac{(12 - 4i - 6i + 2i^2)}{(9 - 3i + 3i - i^2)} - [3 - 5i + 2]$

$= \dfrac{(12 - 2 - 10i)}{(9+1)} - [5 - 5i]$

$= \dfrac{10 - 10i}{10} - 5 + 5i$ 3

$= 1 - i - 5 + 5i$

$\therefore\ z = \underline{\underline{-4 + 4i}}$

b) $z^1 = -2i$; $z^3 = z^2 \times z = (-4) \times (-2i) = \underline{\underline{8i}}$

$z^2 = (-2i)^2 = 4i^2 = \underline{\underline{-4}}$; $z^4 = z^3 \times z = (8i) \times (-2i) = -16i^2 = \underline{\underline{16}}$

(i) $z(z^3 + 8z^2 + 36z + 32) + 128$

$= z^4 + 8z^3 + 36z^2 + 32z + 128$ 2

$= (16) + 8(8i) + 36(-4) + 32(-2i) + 128$

$= 16 + \cancel{64i} - 144 - \cancel{64i} + 128$

$= 0$ $\therefore\ \underline{z = -2i}$ is a root (as $R = 0$)

Q4 (b) (ii) $z = -2i \Rightarrow \bar{z} = 2i$ (As conjugate pair) 1

Thus $(z+2i)=0$ & $(z-2i)=0$

& $(z+2i)$ & $(z-2i)$ are factors of quartic

Multiplying: $(z+2i)(z-2i) = z^2 - 2iz + 2iz - 4i^2$

$= z^2+4$

Need to find remaining quadratic via long division:

$$
\begin{array}{r|l}
 & z^2 + 8z + 32 \\ \hline
z^2+4 & z^4 + 8z^3 + 36z^2 + 32z + 128 \\
 & z^4 \qquad\quad + 4z^2 \\ \hline
 & \qquad 8z^3 + 32z^2 + 32z + 128 \\
 & \qquad 8z^3 \qquad\quad + 32z \\ \hline
 & \qquad\qquad 32z^2 \qquad\quad + 128 \\
 & \qquad\qquad 32z^2 \qquad\quad + 128 \\ \hline
\end{array}
$$

$\Rightarrow z^2+8z+32$ still to solve

$\boxed{a=1;\ b=8;\ c=32}$

$z = \dfrac{-8 \pm \sqrt{8^2 - 4\times1\times32}}{2\times1}$

$= \dfrac{-8 \pm \sqrt{64-128}}{2}$

$= \dfrac{-8 \pm \sqrt{-64}}{2}$

$z = \dfrac{-8 \pm 8i}{2}$

$\therefore z = -4 \pm 4i$

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So 4 Solutions are:

$z_1 = -2i$; $z_2 = 2i$

$z_3 = -4+4i$ & $z_4 = -4-4i$

Q5.

$$\boxed{6^n - 1 \text{ is divisible by } 5 \ \forall n \in \mathbb{N}}$$

Let $n=1$: $6^1 - 1 = 5 = 5 \times 1 \Rightarrow$ true for $n=1$ ✓

Assume true for $n=k$: $6^k - 1 = 5m$, for some positive integer m.

Consider $n=k+1$:

$$\begin{aligned} 6^{k+1} - 1 &= 6^k . 6 - 1 \\ &= 6^k . 6 - 1 (+5 - 5) \\ &= 6^k . 6 - 6 + 5 \\ &= 6(6^k - 1) + 5 \\ &= 6(5m) + 5 \\ &= 5(6m+1) \end{aligned}$$

$\Rightarrow$ multiple of 5, so divisible by 5.

Statement:

As true for $n=1$, assumed true for $n=k$ and by Proof of Mathematical Induction true for $n=k+1$. Therefore true $\forall n \in \mathbb{N}$, and $6^n - 1$ is divisible by 5 $\forall n \in \mathbb{N}$.

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Q6. $x = 10t$; $y = 1 + 12t - t^3$

$$\frac{dx}{dt} = 10 \qquad \frac{dy}{dt} = 12 - 3t^2$$

a) $$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{12-3t^2}{10}$$

Stat Pt when $\frac{dy}{dx} = 0$: $\frac{12-3t^2}{10} = 0$

$12 - 3t^2 = 0$

$12 = 3t^2$

$4 = t^2$

$\therefore t = \pm 2$

Stationary Pts:-

When $t = 2$

$x = 10 \times (2) = 20$

$y = 1 + 12(2) - (2)^3$

$= 1 + 24 - 8$

$= 17$

$\therefore (20, 17)$

When $t = -2$

$x = 10 \times (-2) = -20$

$y = 1 + 12(-2) - (-2)^3$

$= 1 - 24 + 8$

$= -15$

$\therefore (-20, -15)$

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Q6 (b)

$$\boxed{\frac{dx}{dt} = 10 \quad \& \quad \frac{dy}{dx} = \frac{12-3t^2}{10} = \frac{12}{10} - \frac{3}{10}t^2 = \frac{6}{5} - \frac{3t^2}{10}}$$

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}} = \frac{\frac{-6t}{10}}{10} = \frac{-6t}{100} = -\frac{3t}{50}$$

At $t=2$ $\quad \frac{d^2y}{dx^2} = \frac{-3(2)}{50} < 0 \Rightarrow \cap$ Max $(20, 17)$

At $t=-2$ $\quad \frac{d^2y}{dx^2} = \frac{-3(-2)}{50} > 0 \Rightarrow \cup$ Min $(-20, -15)$

3

Q7. $f(x) = ax^3 + bx^2 + cx + 1$

At $(1,0)$ $\Rightarrow$ $0 = a(1)^3 + b(1)^2 + c(1) + 1$
(x, y)

$\therefore \quad a + b + c + 1 = 0$

So $\quad a + b + c = -1 \quad \text{—} (1)$

At $(-2, 9)$ $\Rightarrow$ $9 = a(-2)^3 + b(-2)^2 + c(-2) + 1$
(x, y)

$\therefore \quad 9 = -8a + 4b - 2c + 1$

So $\quad 8a - 4b + 2c = -8$

OR $\quad 4a - 2b + c = -4 \quad \text{—} (2)$

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$\therefore$ these satisfy the systems of equations.

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$$\left.\begin{array}{l} a + b + c = -1 \\ 4a - 2b + c = -4 \\ 12a - 4b + c = 0 \end{array}\right\} \rightarrow \left(\begin{array}{ccc|c} 1 & 1 & 1 & -1 \\ 4 & -2 & 1 & -4 \\ 12 & -4 & 1 & 0 \end{array}\right)\begin{array}{l} r_1 \\ r_2 \\ r_3 \end{array} \rightarrow \begin{array}{r} 12r_1 \\ 3r_2 \\ r_3 \end{array}\left(\begin{array}{ccc|c} 12 & 12 & 12 & -12 \\ 12 & -6 & 3 & -12 \\ 12 & -4 & 1 & 0 \end{array}\right)\begin{array}{l} r_1 \\ r_2 \\ r_3 \end{array}$$

$$\begin{array}{r} \\ r_2 - r_1 \\ r_3 - r_1 \end{array}\left(\begin{array}{ccc|c} 12 & 12 & 12 & -12 \\ 0 & -18 & -9 & 0 \\ 0 & -16 & -11 & 12 \end{array}\right)\begin{array}{l} r_1 \\ r_2 \\ r_3 \end{array} \rightarrow \begin{array}{r} \frac{1}{12}r_1 \\ -\frac{1}{9}r_2 \\ \end{array}\left(\begin{array}{ccc|c} 1 & 1 & 1 & -1 \\ 0 & 2 & 1 & 0 \\ 0 & -16 & -11 & 12 \end{array}\right)\begin{array}{l} r_1 \\ r_2 \\ r_3 \end{array}$$

$$\begin{array}{r} r_1 \\ 8r_2 \\ r_3 \end{array}\left(\begin{array}{ccc|c} 1 & 1 & 1 & -1 \\ 0 & 16 & 8 & 0 \\ 0 & -16 & -11 & 12 \end{array}\right)\begin{array}{l} r_1 \\ r_2 \\ r_3 \end{array} \rightarrow \begin{array}{r} \\ \\ r_3 + r_2 \end{array}\left(\begin{array}{ccc|c} 1 & 1 & 1 & -1 \\ 0 & 16 & 8 & 0 \\ 0 & 0 & -3 & 12 \end{array}\right)$$

Solution

$-3c = 12$

$c = -4$

$16b + 8c = 0$

$16b - 32 = 0$

$16b = 32$

$b = 2$

$a + b + c = -1$

$a + 2 - 4 = -1$

$a = 1$

$\therefore f(x) = x^3 + 2x^2 - 4x + 1$

Q8. $g(x) = x - 2 + \dfrac{9}{(x+2)}$

(a) (i) Vertical Asymptote: $x = -2$

& Non-Vertical Asymptote: $y = x - 2$ 2

(ii) $x = 0$: $y = 0 - 2 + \dfrac{9}{(0+2)} = -2 + \dfrac{9}{2} = \dfrac{5}{2}$

$\therefore$ Crosses y-axis at $(0, 5/2)$ 1

(b) $g(x) = x - 2 + 9(x+2)^{-1}$

$g'(x) = 1 - 9(x+2)^{-2}$

$= 1 - \dfrac{9}{(x+2)^2}$

Stat Pts $g'(x) = 0$:- $1 - \dfrac{9}{(x+2)^2} = 0$

$1 = \dfrac{9}{(x+2)^2}$

$(x+2)^2 = 9$ 5

$x + 2 = \pm 3$

$\therefore x = -2 \pm 3$

So $x = -5$ & $x = 1$

$x = -5$: $y = -5 - 2 + \dfrac{9}{(-5+2)} = -7 + \dfrac{9}{(-3)} = -7 - 3 = -10 \Rightarrow (-5, -10)$

$x = 1$: $y = 1 - 2 + \dfrac{9}{(1+2)} = -1 + 9/(3) = -1 + 3 = 2 \Rightarrow (1, 2)$

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$g(x) = x - 2 + 9(x+2)^{-1}$

$g'(x) = 1 - 9(x+2)^{-2}$

$$\boxed{g''(x) = 18(x+2)^{-3} = \frac{18}{(x+2)^3}}$$

Alternatively use a nature table

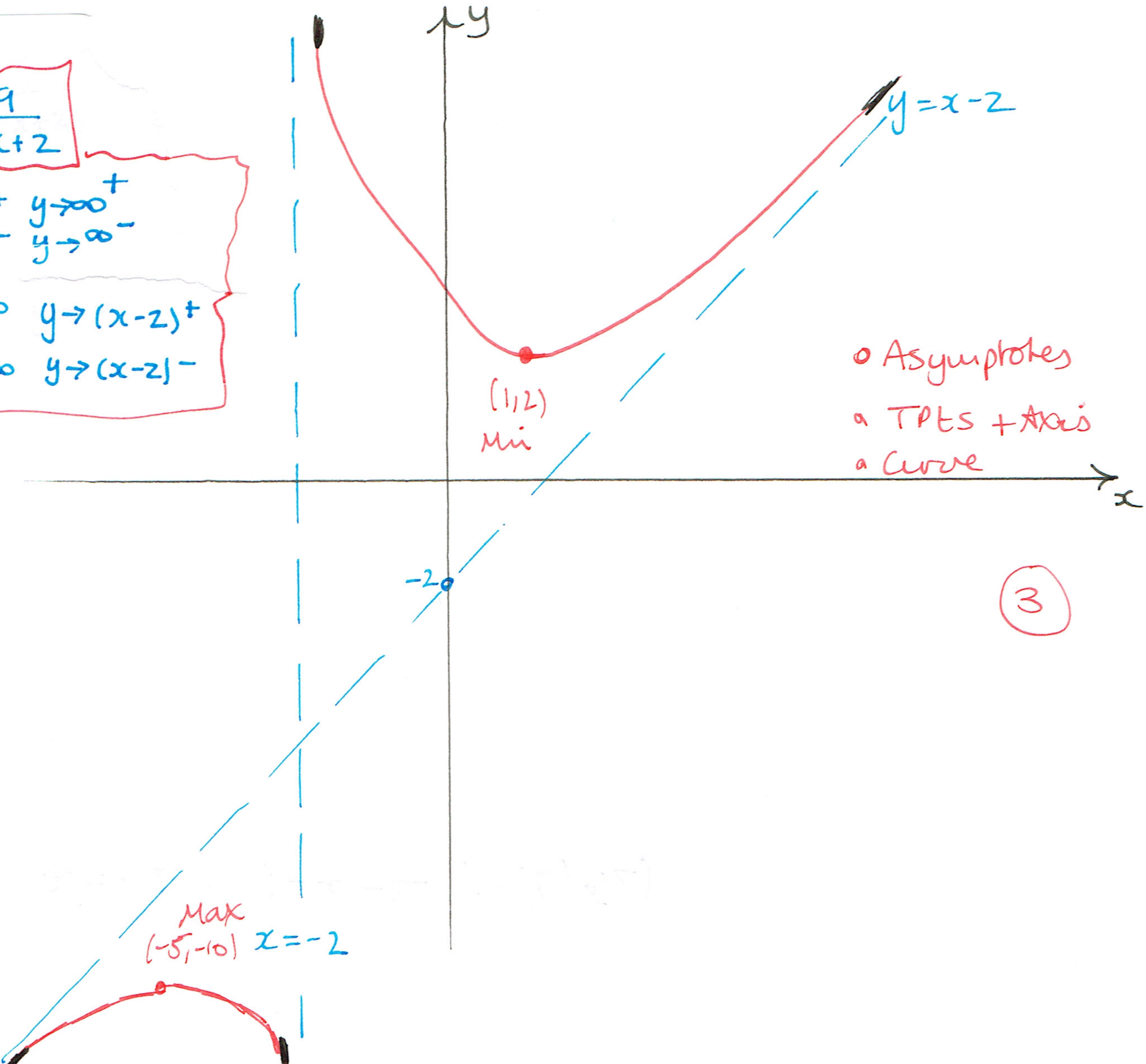
At $x = -5$: $g''(-5) = \dfrac{18}{(-5+2)^3} = \dfrac{18}{(-3)^3} = \dfrac{18}{-27} < 0$ $\cap$ MIN $(-5,-10)$

At $x = 1$: $g''(1) = \dfrac{18}{(1+2)^3} = \dfrac{18}{(3)^3} = \dfrac{18}{27} > 0$ $\cup$ Max $(1,2)$

$x - 2 + \dfrac{9}{x+2}$

As $x \to -2^+$ $y \to \infty^+$
$x \to -2^-$ $y \to \infty^-$

As $x \to \infty$ $y \to (x-2)^+$
$x \to -\infty$ $y \to (x-2)^-$



- Asymptotes
- TPts + Axis
- Curve

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Q9. $\boxed{S_n = n^2 - 6n \qquad S_1 = U_1 = a = 1^2 - 6 = -5}$

$S_2 = 2^2 - 6(2) = 4 - 12 = \underline{\underline{-8}}$

$S_2 = U_1 + U_2$, So $-8 = -5 + U_2$

$-3 = U_2$

$\therefore \underline{\underline{U_2 = -3}}$

$\boxed{\text{Sequence: } -5, -3, \ldots \text{ with } a = -5 \text{ \& } d = 2}$

$U_n = a + (n-1)d$

$U_n = -5 + (n-1) \times 2$ 3

$U_n = 2n - 2 - 5$

$\underline{\underline{U_n = 2n - 7}}$

(b) $\underline{a = 64; r = \frac{-32}{64} = \frac{-1}{2}}$

$U_n = ar^{n-1}$

$\frac{1}{4} = 64\left(\frac{-1}{2}\right)^{n-1}$

$\frac{1}{256} = \left(\frac{-1}{2}\right)^{n-1}$

$\frac{1}{2^8} = \left(\frac{-1}{2}\right)^8 = \left(\frac{-1}{2}\right)^{n-1}$

$\therefore \quad 8 = n - 1$

$\Rightarrow \quad \underline{\underline{n = 9}}$ 3

$S_n = \frac{a(1-r^n)}{1-r} \Rightarrow S_9 = \frac{64\left(1 - \left(-\frac{1}{2}\right)^9\right)}{1 - \left(-\frac{1}{2}\right)} = \frac{64\left(1 + \frac{1}{512}\right)}{\frac{3}{2}}$

$\therefore S_9 = \frac{513}{12}$ or $\underline{42.75}$

Q10. AH 2009/10 Units 1/2 Prelim

$$\int_0^{2/5} \sqrt{16-25x^2}\,dx$$

$$= \int_0^{2/5} \sqrt{16-(5x)^2}\,dx$$

$$= \int_{\pi/2}^{\pi/3} \sqrt{16-(4\cos\theta)^2}\cdot\left(-\frac{4}{5}\sin\theta\,d\theta\right)$$

$$= \int_{\pi/2}^{\pi/3} \sqrt{16-16\cos^2\theta}\cdot\left(-\frac{4}{5}\sin\theta\,d\theta\right)$$

$$= -\frac{4}{5}\int_{\pi/2}^{\pi/3} \sqrt{16(1-\cos^2\theta)}\cdot\sin\theta\,d\theta$$

$$= -\frac{4}{5}\int_{\pi/2}^{\pi/3} 4\underbrace{\sqrt{\sin^2\theta}\cdot\sin\theta}\,d\theta$$

$$= -\frac{16}{5}\int_{\pi/2}^{\pi/3} \sin^2\theta\,d\theta$$

$$= -\frac{16}{5}\int_{\pi/2}^{\pi/3} \left(\frac{1-\cos 2\theta}{2}\right)d\theta$$

$$= -\frac{8}{5}\int_{\pi/2}^{\pi/3} (1-\cos 2\theta)\,d\theta$$

$$5x = 4\cos\theta$$

$$x = \frac{4}{5}\cos\theta$$

$$\frac{dx}{d\theta} = -\frac{4}{5}\sin\theta$$

$$\therefore \boxed{dx = -\frac{4}{5}\sin\theta\,d\theta}$$

$\underline{x=0}$: $0 = \frac{4}{5}\cos\theta$

$\cos\theta = 0$

$\underline{\underline{\theta = \pi/2}}$

$\underline{x = \frac{2}{5}}$: $\frac{2}{5} = \frac{4}{5}\cos\theta$

$\cos\theta = 1/2$

$\underline{\underline{\theta = \pi/3}}$

$$\cos 2\theta = 1 - 2\sin^2\theta$$

$$2\sin^2\theta = 1-\cos 2\theta$$

$$\therefore \sin^2\theta = \frac{1-\cos 2\theta}{2}$$

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$$= -\frac{8}{5}\left[\theta - \frac{\sin 2\theta}{2}\right]_{\pi/2}^{\pi/3} = -\frac{8}{5}\left[\left(\frac{\pi}{3} - \frac{\sin\left(\frac{2\pi}{3}\right)}{2}\right) - \left(\frac{\pi}{2} - \frac{\sin(\pi)}{2}\right)\right]$$

$$= -\frac{8}{5}\left(\frac{\pi}{3} - \frac{\sqrt{3}/2}{2} - \frac{\pi}{2} + 0\right) = -\frac{8}{5}\left(\frac{\pi}{3} - \frac{\pi}{2} - \frac{\sqrt{3}}{4}\right)$$

$$= -\frac{8}{5}\left(\frac{4\pi - 6\pi - 3\sqrt{3}}{12}\right) = \frac{-2(-2\pi - 3\sqrt{3})}{5\ (3)} = \underline{\frac{4\pi + 6\sqrt{3}}{15}}$$