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Q1. $R = P - 2Q = \begin{pmatrix} 6 & -3 \\ 2 & -5 \end{pmatrix} - 2\begin{pmatrix} 1 & -1 \\ 2 & -3 \end{pmatrix}$

$$= \begin{pmatrix} 6 & -3 \\ 2 & -5 \end{pmatrix} + \begin{pmatrix} -2 & 2 \\ -4 & 6 \end{pmatrix}$$

$$= \begin{pmatrix} 4 & -1 \\ -2 & 1 \end{pmatrix}$$

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$$R^{-1} = \frac{1}{(4-2)}\begin{pmatrix} 1 & 1 \\ 2 & 4 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 2 & 4 \end{pmatrix} = \begin{pmatrix} 1/2 & 1/2 \\ 1 & 2 \end{pmatrix}$$

Q2. $x_{n+1} = \frac{1}{5}\left\{4x_n - \frac{27}{x_n^2}\right\}$

Let $x_n, x_{n+1}, \dots$ be represented by λ

$$\boxed{\lambda = \tfrac{1}{5}\left(4\lambda - \frac{27}{\lambda^2}\right)}$$

$$5\lambda = 4\lambda - \frac{27}{\lambda^2}$$

$$\lambda = \frac{-27}{\lambda^2}$$

$$\lambda^3 = -27$$

3

$$\therefore \quad \lambda = -3$$

$\therefore$ fixed point is $x = -3$

Q3. $L_1 : \frac{x-1}{3} = \frac{y+1}{4} = \frac{z-1}{-2} \quad (=\lambda) \Rightarrow$

$$\boxed{\begin{aligned} L_1 : x &= 1+3\lambda \\ y &= -1+4\lambda \\ z &= 1-2\lambda \end{aligned}}$$

$\pi : \; 2x - y - 4z = 9$

$$2x - y - 4z = 9$$
$$2(1+3\lambda) - (-1+4\lambda) - 4(1-2\lambda) = 9$$
$$2 + 6\lambda + 1 - 4\lambda - 4 + 8\lambda = 9$$
$$10\lambda - 1 = 9$$
$$10\lambda = 10$$
$$\therefore \underline{\underline{\lambda = 1}}$$

4

L_1: $x = 1+3\lambda = 1+3 = 4$
$y = -1+4\lambda = -1+4 = 3$
$z = 1-2\lambda = 1-2 = -1$

$\Rightarrow$ Plane & line intersect at $T(4,3,-1)$

b) $\underline{d} = (3,4,-2)$ & $\underline{n} = (2,-1,-4)$

$$\cos\theta = \frac{\underline{d}.\underline{n}}{|\underline{d}||\underline{n}|} = \frac{\begin{pmatrix}3\\4\\-2\end{pmatrix}\cdot\begin{pmatrix}2\\-1\\-4\end{pmatrix}}{\sqrt{9+16+4}\,\sqrt{4+1+16}} = \frac{6-4+8}{\sqrt{29}\sqrt{21}} = \frac{10}{\sqrt{609}}$$

$$\theta = \cos^{-1}\left(\frac{10}{\sqrt{609}}\right)$$

$$\theta = 66.1°$$

4

* Angle between line & Plane is COMPLEMENT of θ ie. $(90-\theta)$

$\rightarrow$ Angle $= 90 - 66.1$
$= 23.9°$

Q3 (c) π_2, α, parallel to π_1 $\Rightarrow$ Same normal $(2,-1,-4)$

L: $x = 1+3\lambda$ meets $R(-5,-9,5)$
$y = -1+4\lambda$ on plane α.
$z = 1-2\lambda$

Plane α: $2x - y - 4z = k$
$2(-5) - (-9) - 4(5) = k$
$-10 + 9 - 20 = k$
$\therefore k = -21$

$\Rightarrow$ Plane α
$2x - y - 4z = -21$

3

Q4. a) $e^{\int \frac{\sin x}{\cos x} dx} = e^{\int \frac{\sin x}{u} \cdot \frac{du}{-\sin x}}$

Let $u = \cos x$
$\frac{du}{dx} = -\sin x$
$\frac{du}{-\sin x} = dx$

$= e^{\int \frac{du}{-u}}$

$= e^{-\ln u}$

$= e^{\ln(u^{-1})}$

$= u^{-1}$

$= \frac{1}{u}$

$= \frac{1}{\cos x}$

$= \sec x$, as required.

2

Q4(b) (i) $\cos x \dfrac{dy}{dx} + (\sin x)y = 2\cos^3 x \sin x - 1$

$$\frac{dy}{dx} + \left(\frac{\sin x}{\cos x}\right)y = 2\cos^2 x \sin x - \frac{1}{\cos x}$$

$$\boxed{I(x) = e^{\int P(x)dx} = e^{\int \frac{\sin x}{\cos x}dx} = \sec x}$$

× by $I(x)$: $\sec x\left[\dfrac{dy}{dx} + \left(\dfrac{\sin x}{\cos x}\right)y\right] = \sec x\left[2\cos^2 x \sin x - \dfrac{1}{\cos x}\right]$

$$\boxed{\frac{d}{dx}\left[I(x)y\right] = I(x)f(x)}$$

$$\frac{d}{dx}(\sec x\, y)dx = \sec x\left[2\cos^2 x \sin x - \frac{1}{\cos x}\right]$$

$$\int \frac{d}{dx}\left(\frac{1}{\cos x}\cdot y\right)\cdot dx = \int \frac{1}{\cos x}\left(2\cos^2 x \sin x - \frac{1}{\cos x}\right)\cdot dx$$

$$\frac{y}{\cos x} = \int \left(2\cos x \sin x - \frac{1}{\cos^2 x}\right)dx$$

$$\frac{y}{\cos x} = \int (\sin 2x - \sec^2 x)\,dx$$

6

$$\frac{y}{\cos x} = -\frac{\cos 2x}{2} - \tan x + c$$

b)(ii)

General Sol: $y = \cos x\left(-\frac{1}{2}\cos 2x - \tan x + c\right)$

$\boxed{\begin{array}{l} x = \pi/4 \\ y = 3\sqrt{2} \end{array}} \Rightarrow$ $3\sqrt{2} = \cos\left(\frac{\pi}{4}\right)\left(-\frac{1}{2}\cos\left(\frac{\pi}{2}\right) - \tan\left(\frac{\pi}{4}\right) + c\right)$

$$3\sqrt{2} = \frac{1}{\sqrt{2}}(0 - 1 + c)$$

2

$$6 = -1 + c \implies c = 7$$

$\therefore$ P.Sol is $y = \cos x\left(-\frac{1}{2}\cos 2x - \tan x + 7\right)$

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Q5. $\sum_{r=1}^{n} \frac{3}{(3r-1)(3r+2)} = \frac{1}{2} - \frac{1}{3n+2}$

Let $n=1$ $\quad LHS = \frac{3}{(3-1)(3+2)} = \frac{3}{2\times 5} = \frac{3}{10}$

$RHS = \frac{1}{2} - \frac{1}{(3+2)} = \frac{1}{2} - \frac{1}{5} = \frac{5-2}{10} = \frac{3}{10}$

LHS = RHS ✓ true for $n=1$

Assume true for $n=k$ $\quad \sum_{r=1}^{n=k} \frac{3}{(3r-1)(3r+2)} = \frac{1}{2} - \frac{1}{3k+2}$

Consider $n=k+1$

$$\sum_{r=1}^{n=k} \frac{3}{(3r-1)(3r+2)} + \boxed{\frac{3}{(3(k+1)-1)(3(k+1)+2)}} = \left(\frac{1}{2} - \frac{1}{3k+2}\right) + \boxed{\frac{3}{(3(k+1)-1)(3(k+1)+2)}}$$

$$= \left(\frac{1}{2} - \frac{1}{3k+2}\right) + \frac{3}{(3k+3-1)(3k+3+2)}$$

$$= \frac{1}{2} - \frac{1}{3k+2} + \frac{3}{(3k+2)(3k+5)}$$

$$= \frac{1}{2} + \frac{3}{(3k+2)(3k+5)} - \frac{1.(3k+5)}{(3k+2)(3k+5)}$$

$$= \frac{1}{2} + \frac{3-3k-5}{(3k+2)(3k+5)}$$

5

$$= \frac{1}{2} + \frac{-3k-2}{(3k+2)(3k+5)}$$

$$= \frac{1}{2} + \frac{-\cancel{(3k+2)}}{\cancel{(3k+2)}(3k+5)}$$

$$= \frac{1}{2} - \frac{1}{(3k+5)}$$

$$= \frac{1}{2} - \frac{1}{3(k+1)+2} \quad \text{as required}$$

As true for $n=1$, assumed true for $n=k$ and by proof of mathematical Induction also true for $n=k+1$, conjecture is true $\forall n \in \mathbb{N}$.

Q5. $$\boxed{\sum_{r=1}^{n} \frac{3}{(3r-1)(3r+2)} = \frac{1}{2} - \frac{1}{3n+2}}$$

As $n \to \infty$ Limit tends to: $\frac{1}{2} - \frac{1}{3\infty+2}$

$$= \frac{1}{2} - 0 \quad \left(\frac{1}{\infty} \to 0\right)$$

1

$$\therefore \lim_{n\to\infty} = \frac{1}{2}$$

Q6. $271 = 45 \cdot 6 + 1$

$45 = 7 \cdot 6 + 3$

$7 = 1 \cdot 6 + 1$

$1 = 0 \cdot 6 + 1$

3

$\therefore 271 = 1131_6$

Q7.

$f(x) = \ln(1+x)$

$f'(x) = \frac{1}{1+x} = (1+x)^{-1}$

$f''(x) = -(1+x)^{-2} = \frac{-1}{(1+x)^2}$

$f'''(x) = 2(1+x)^{-3} = \frac{2}{(1+x)^3}$

$f^{iv}(x) = -6(1+x)^{-4} = \frac{-6}{(1+x)^4}$

$f(0) = \ln(1) = 0$

$f'(0) = \frac{1}{1} = 1$

$f''(0) = \frac{-1}{1} = -1$

$f'''(0) = \frac{2}{1} = 2$

$f^{iv}(0) = \frac{-6}{1} = -6$

$$\boxed{f(x) \simeq f(0) + f'(0)\frac{x}{1!} + f''(0)\frac{x^2}{2!} + f'''(0)\frac{x^3}{3!} + \dots + \frac{f^{n}(0)\,x^n}{n!}}$$

$$\ln(1+x) \simeq 0 + \frac{x}{1!} - \frac{x^2}{2!} + \frac{2x^3}{3!} - \frac{6x^4}{4!} + \dots$$

$$= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots$$

$$\ln(\cos x + x\cos x) = \ln(\cos x(1+x))$$

$$= \ln(\cos x) + \ln(1+x)$$

$$= \left(-\frac{x^2}{2} - \frac{x^4}{12}\right) + \left(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4}\right) + \dots$$

$$= x - x^2 + \frac{x^3}{3} - \frac{x^4}{12} - \frac{3x^4}{12} + \dots$$

6

$$= x - x^2 + \frac{x^3}{3} - \frac{4x^4}{12} + \dots$$

$$\therefore \ln(\cos x + x\cos x) = x - x^2 + \frac{x^3}{3} - \frac{x^4}{3} + \dots$$

Q8.

(a) $AB = \begin{pmatrix} 1 & 1 & -1 \\ -1 & 0 & 2 \\ 1 & 2 & -1 \end{pmatrix} \begin{pmatrix} 4 & 1 & -2 \\ -1 & 0 & 1 \\ 2 & 1 & -1 \end{pmatrix}$

$= \begin{pmatrix} 4-1-2 & 1\,0-1 & -2+1+1 \\ -4+0+4 & -1+0+2 & 2+0-2 \\ 4-2-2 & 1+0-1 & -2+2+1 \end{pmatrix}$

$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

1

$= I \quad \Rightarrow$ A is the INVERSE of B & B is the INVERSE of A

(b) $\begin{pmatrix} 4 & 1 & -2 \\ -1 & 0 & 1 \\ 2 & 1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ 5 \end{pmatrix}$

$B\,.(X) = C$

$A\,(B\,x) = AC$

$I\,x = AC$

$\therefore \quad x = AC$

To solve X multiply both by A to get Inverse, I.

$AC = \begin{pmatrix} 1 & 1 & -1 \\ -1 & 0 & 2 \\ 1 & 2 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ -2 \\ 5 \end{pmatrix} = \begin{pmatrix} 1-2-5 \\ -1+0+10 \\ 1-4-5 \end{pmatrix} = \begin{pmatrix} -6 \\ 9 \\ -8 \end{pmatrix}$

3

$\Rightarrow \quad x = -6\,; \quad y = 9 \quad \& \quad z = -8$

Q9.

$$4\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + y = 3x+4$$

Aux Eqn $\quad 4m^2+4m+1=0$

$(2m+1)^2 = 0$

$\therefore\ m = -\frac{1}{2}$ (twice)

$\therefore\ y_c = Ae^{-1/2x} + Bxe^{-1/2x}$

Particular Integral $\quad$ Let $y = ax+b$

Then $\frac{dy_p}{dx} = a$ & $\frac{d^2y_p}{dx^2} = 0$

$$4\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + y = 3x+4$$

$$4(0) + 4(a) + (ax+b) = 3x+4$$

$4a + ax + b = 3x+4$

6

$ax + (4a+b) = 3x+4$

$\therefore\ ax = 3x$ & $4a+b=4$

$a = 3 \qquad 12+b=4$

$\therefore\ b = -8$

$\Rightarrow y_p = 3x-8$

General Solution:

$\therefore\ y = y_c + y_p = Ae^{-1/2x} + Bxe^{-1/2x} + 3x - 8$

Q9 GS $y = Ae^{-1/2x} + Bxe^{-1/2x} + 3x - 8$

Remembe Product Rule

$$\frac{dy}{dx} = -\frac{1}{2}Ae^{-1/2x} + \left(Be^{-1/2x} - \frac{1}{2}Bxe^{-1/2x}\right) + 3$$

$$\frac{d^2y}{dx^2} = \frac{1}{4}Ae^{-1/2x} - \frac{1}{2}Be^{-1/2x} + \left(\frac{1}{4}Bxe^{-1/2x} - \frac{1}{2}Be^{-1/2x}\right)$$

$x = 0;\ \frac{dy}{dx} = -3$:

$$-3 = -\frac{1}{2}Ae^0 + Be^0 - 0 + 3$$

$$-3 = -\tfrac{1}{2}A + B + 3$$

$$-6 = -\frac{1}{2}A + B$$

$$-12 = -A + 2B \Rightarrow A - 2B = 12 \quad -(1)$$

$x = 0;\ \frac{d^2y}{dx^2} = 4$:

$$4 = \frac{1}{4}Ae^0 - \frac{1}{2}Be^0 + 0 - \frac{1}{2}Be^0$$

$$4 = \frac{1}{4}A - B$$

$$16 = A - 4B \Rightarrow A - 4B = 16 \quad -(2)$$

$A - 2B = 12 \quad -(1)$

$A - 4B = 16 \quad -(2)$

(1) - (2): $2B = -4$

$B = -2$

Subst into (1) $B = -2$

$A - 2B = 12$

$A + 4 = 12$

$\therefore A = 8$

4

Partialar Solution: $y = 8e^{-1/2x} - 2xe^{-1/2x} + 3x - 8$