Inverse Differentiation

Exercise 1

Q1. a) $f(x) = \tan^{-1}(3x)$

$$f'(x) = \frac{1}{1+(3x)^2} \cdot 3$$

$$= \frac{3}{1+9x^2}$$

b) $f(x) = \cos^{-1}(4x)$

$$f'(x) = \frac{-1}{\sqrt{1-(4x)^2}} \cdot 4$$

$$= \frac{-4}{\sqrt{1-16x^2}}$$

c) $f(x) = \sin^{-1}\left(\frac{x}{3}\right)$

$$f'(x) = \frac{1}{\sqrt{1-\left(\frac{x}{3}\right)^2}} \cdot \frac{1}{3}$$

$$= \frac{1}{3\sqrt{1-\frac{x^2}{9}}}$$

$$= \frac{1}{3\sqrt{\frac{9-x^2}{9}}}$$

$$= \frac{1}{\frac{\cancel{3}\sqrt{9-x^2}}{\cancel{\sqrt{9}}}}$$

$$= \frac{1}{\sqrt{9-x^2}}$$

d) $f(x) = \tan^{-1}\left(\frac{x}{2}\right)$

$$f'(x) = \frac{1}{1+\left(\frac{x}{2}\right)^2} \cdot \frac{1}{2}$$

$$= \frac{1}{2\left(1+\frac{x^2}{4}\right)}$$

$$= \frac{1}{2\left(\frac{4+x^2}{4}\right)}$$

$$= \frac{4}{2(4+x^2)}$$

$$= \frac{2}{4+x^2}$$

Inverse Trig Differentiation

Q1. (e) $f(x) = \cos^{-1}\left(\frac{1}{x}\right)$

$= \cos^{-1}(x^{-1})$

$$f'(x) = \frac{-1}{\sqrt{1-(1/x)^2}} \cdot -x^{-2}$$

$$= \frac{-1}{\sqrt{1-\frac{1}{x^2}}} \cdot \frac{-1}{x^2}$$

$$= \frac{1}{x^2\sqrt{\frac{x^2-1}{x^2}}}$$

$$= \frac{1}{x^2\frac{\sqrt{x^2-1}}{x}}$$

$$= \frac{1}{x\sqrt{x^2-1}}$$

(f) $f(x) = \sin^{-1}(\cos x)$

$$f'(x) = \frac{1}{\sqrt{1-(\cos x)^2}} \cdot -\sin x$$

$$= \frac{-\sin x}{\sqrt{1-\cos^2 x}}$$

$$= \frac{-\sin x}{\sqrt{\sin^2 x}} \quad *$$

$$= \frac{-\sin x}{\sin x}$$

$$= -1$$

$*$ $\sin^2 x + \cos^2 x = 1 \Rightarrow$
$\sin^2 x = 1 - \cos^2 x$

Implicit Differentiation

Q2. (a) $x^2+y^2=6$

$$2x+2y\frac{dy}{dx}=0$$

$$2y\frac{dy}{dx}=-2x$$

$$\frac{dy}{dx}=\frac{-2x}{2y}$$

$$\therefore \quad \frac{dy}{dx}=\frac{-x}{y}$$

(b) $x^2-y^2=6$

$$2x-2y\frac{dy}{dx}=0$$

$$2x=2y\frac{dy}{dx}$$

$$\frac{2x}{2y}=\frac{dy}{dx}$$

$$\therefore \quad \frac{dy}{dx}=\frac{x}{y}$$

(c) $2x^2+3y^2=6$

$$4x+6y\frac{dy}{dx}=0$$

$$6y\frac{dy}{dx}=-4x$$

$$\frac{dy}{dx}=\frac{-4x}{6y}$$

$$\therefore \quad \frac{dy}{dx}=\frac{-2x}{3y}$$

(d) $2x^2+3y^2=6x$

$$4x+6y\frac{dy}{dx}=6$$

$$6y\frac{dy}{dx}=6-4x$$

$$\frac{dy}{dx}=\frac{6-4x}{6y}$$

$$\therefore \quad \frac{dy}{dx}=\frac{3-2x}{3y}$$

Implicit Differentiation

Q2(e) $2x^2+y^2=xy$

$$4x+2y\frac{dy}{dx}=y+x\frac{dy}{dx}$$

$$(2y-x)\frac{dy}{dx}=y-4x$$

$$\frac{dy}{dx}=\frac{y-4x}{2y-x}$$

$u=x \quad v=y$

$u'=1 \quad v'=\frac{dy}{dx}$

$\boxed{u'v+uv'}$

$1.y+x\frac{dy}{dx}$

Q2(f) $xy=3$

$$y+x\frac{dy}{dx}=0$$

$$x\frac{dy}{dx}=-y$$

$$\frac{dy}{dx}=-\frac{y}{x}$$

$u=x \quad v=y$

$u'=1 \quad v'=\frac{dy}{dx}$

$\boxed{u'v+uv'}$

$y+x\frac{dy}{dx}$

Q2(g) $xy^2=9$

$$y^2+2xy\frac{dy}{dx}=0$$

$$2xy\frac{dy}{dx}=-y^2$$

$$\frac{dy}{dx}=\frac{-y^2}{2xy}=\frac{-y}{2x}$$

$u=x \quad v=y^2$

$u'=1 \quad v'=2y\frac{dy}{dx}$

$\boxed{u'v+uv'}$

$y^2+2xy\frac{dy}{dx}$

Implicit Differentiation

Q2 (h) $x^3y^2 = 3$

$$3x^2y^2 + 2x^3y\frac{dy}{dx} = 0$$

$$2x^3y\,\frac{dy}{dx} = -3x^2y^2$$

$$\frac{dy}{dx} = \frac{-3x^2y^2}{2x^3y}$$

$$\therefore \frac{dy}{dx} = \frac{-3y}{2x}$$

$u = x^3$ $\quad v = y^2$

$u' = 3x^2$ $\quad v' = 2y\frac{dy}{dx}$

$\boxed{u'v + uv'}$

$3x^2y^2 + 2x^3y\frac{dy}{dx}$

Q2 (i) $x^2 + \underbrace{3xy}_{} + y^2 = 2$

$$2x + \left(3y + 3x\frac{dy}{dx}\right) + 2y\frac{dy}{dx} = 0$$

$$(3x + 2y)\frac{dy}{dx} = -2x - 3y$$

$$\therefore \frac{dy}{dx} = \frac{-2x - 3y}{3x + 2y}$$

$u = 3x$ $\quad v = y$

$u' = 3$ $\quad v' = \frac{dy}{dx}$

$\boxed{u'v + uv'}$

$\left(3y + 3x\frac{dy}{dx}\right)$

Implicit Differentiation

Q3. (a) $x^2 + xy + 4y^2 = 16$ at $(3,1)$

$$2x + y + x\frac{dy}{dx} + 8y\frac{dy}{dx} = 0$$

$$(x+8y)\frac{dy}{dx} = -2x - y$$

$$\frac{dy}{dx} = \frac{-2x-y}{x+8y}$$

At $(3,1)$ $\quad m = \dfrac{-2(3)-(1)}{(3)+8(1)} = \dfrac{-6-1}{3+8} = \dfrac{-7}{11}$

$$(y-b) = m(x-a)$$

$$(y-1) = \frac{-7}{11}(x-3)$$

$$11y - 11 = -7x + 21$$

$7x + 11y - 32 = 0$ is equation of tangent at $(3,1)$

Implicit + Tangents

Q3(b) $x^3+y^3=4y^2$ at $(2,2)$

$$3x^2+3y^2\frac{dy}{dx}=8y\frac{dy}{dx}$$

$$(3y^2-8y)\frac{dy}{dx}=-3x^2$$

$$\frac{dy}{dx}=\frac{-3x^2}{3y^2-8y} \text{ or } \frac{-3x^2}{y(3y-8)}$$

At $(2,2)$: $m=\frac{-3(2)^2}{2(3(2)-8)}=\frac{-3\times4}{2(6-8)}=\frac{-12}{-4}=3$

$$(y-b)=m(x-a)$$

$$(y-2)=3(x-2)$$

$$y-2=3x-6$$

$$y=3x-4$$

OR $3x-y-4=0$ in General Equation format

Implicit + Tangents

Q3(c) $y(x+y)^2 = 3(x^3-5)$

$y(x^2+2xy+y^2) = 3x^3-15$

$\underbrace{x^2y}_{①} + 2x\underbrace{y^2}_{②} + y^3 = 3x^3-15$

$\left(2xy + x^2\frac{dy}{dx}\right) + \left(2y^2 + 4xy\frac{dy}{dx}\right) + 3y^2\frac{dy}{dx} = 9x^2$

$(x^2+4xy+3y^2)\frac{dy}{dx} = 9x^2-2xy-2y^2$

$\therefore \frac{dy}{dx} = \frac{9x^2-2xy-2y^2}{x^2+4xy+3y^2}$

① $\int \boxed{x^2y}$

$u = x^2 \quad v = y$

$u' = 2x \quad v' = \frac{dy}{dx}$

$\boxed{u'v+uv'}$

$\left(2xy + x^2\frac{dy}{dx}\right)$

② $\boxed{2xy^2}$

$u = 2x \quad v = y^2$

$u' = 2 \quad v' = 2y\frac{dy}{dx}$

$\boxed{u'v+uv'}$

$\left(2y^2 + 4xy\frac{dy}{dx}\right)$

At $(2,1)$: $m = \frac{9(2)^2-2(2)(1)-2(1)^2}{(2)^2+4\times(2)\times(1)+3(1)^2}$

$m = \frac{9\times4-4-2}{4+8+3} = \frac{36-6}{15} = \frac{30}{15} = 2$

$(y-b) = m(x-a)$

$(y-1) = 2(x-2)$

$y-1 = 2x-4$

$y = 2x-3$

OR $2x-y-3 = 0$

Implicit + Tangents

Q3. $xy(x+y)=84$

(d) $\underbrace{x^2y}_{①} + \underbrace{xy^2}_{②} = 84$

$$\left(2xy + x^2\frac{dy}{dx}\right) + \left(y^2 + 2xy\frac{dy}{dx}\right) = 0$$

$$(x^2+2xy)\frac{dy}{dx} = -2xy - y^2$$

$$\therefore \frac{dy}{dx} = \frac{-y(y+2x)}{x(x+2y)}$$

① $\boxed{x^2y}$

$u = x^2 \quad v = y$

$u' = 2x \quad v' = dy/dx$

$\boxed{u'v + uv'}$

$\left(2xy + x^2\frac{dy}{dx}\right)$

② $\boxed{xy^2}$

$u = x \quad v = y^2$

$u' = 1 \quad v' = 2y\frac{dy}{dx}$

$\boxed{u'v + uv'}$

$\left(y^2 + 2xy\frac{dy}{dx}\right)$

At (3,4)

$$m = \frac{-4(4+2(3))}{3(3+2(4))}$$

$$m = \frac{-4(4+6)}{3(3+8)} = \frac{-4(10)}{3(11)} = \frac{-40}{33}$$

$$(y-b) = m(x-a)$$

$$(y-4) = -\frac{40}{33}(x-3)$$

$$33y - 132 = -40x + 120$$

$$40x + 33y - 252 = 0$$

1. (a) $f(x) = x\cos^{-1}(x)$

$$f'(x) = u'v + uv'$$
$$= \cos^{-1}(x) - \frac{x}{\sqrt{1-x^2}}$$

$u = x \quad v = \cos^{-1}(x)$
$u' = 1 \quad v' = \frac{-1}{\sqrt{1-x^2}}$

(b) $f(x) = (1-x^2)\sin^{-1}(x)$

$$f'(x) = -2x\sin^{-1}(x) + \frac{(1-x^2)}{\sqrt{1-x^2}}$$
$$= -2x\sin^{-1}(x) + \sqrt{1-x^2}$$

$u = (1-x^2) \quad v = \sin^{-1}(x)$
$u' = -2x \quad v' = \frac{1}{\sqrt{1-x^2}}$

(c) $f(x) = (1+x)\tan^{-1}(x)$

$$f'(x) = \tan^{-1}(x) + \frac{1+x}{(1+x^2)}$$

$u = 1+x \quad v = \tan^{-1}(x)$
$u' = 1 \quad v' = \frac{1}{(1+x^2)}$

(d) $f(x) = \frac{1}{x}\tan^{-1}\left(\frac{1}{x}\right)$

$$f'(x) = \frac{-1}{x^2}\tan^{-1}\left(\frac{1}{x}\right) + \frac{1}{x}\left(\frac{-1}{x^2+1}\right)$$
$$= \frac{-\tan^{-1}(1/x)}{x^2} + \frac{-1}{x(x^2+1)}$$
$$= \frac{(x^2+1)\left(-\tan^{-1}\left(\frac{1}{x}\right)\right) + -1(x)}{x^2(x^2+1)}$$
$$= \frac{-\left(x + (x^2+1)\tan^{-1}(x)\right)}{x^2(x^2+1)}$$

$u = x^{-1} \quad v = \tan^{-1}\left(\frac{1}{x}\right)$
$u' = -x^{-2} = \frac{-1}{x^2}$
$v' = \frac{1}{1+\left(\frac{1}{x}\right)^2} \cdot \frac{-1}{x^2}$
$= \frac{1}{\left(1+\frac{1}{x^2}\right)} \cdot \frac{-1}{x^2}$
$= \frac{-1}{\cancel{x^2}\left(\frac{x^2+1}{\cancel{x^2}}\right)}$
$= \frac{-1}{x^2+1}$

Ex 6 Inverse Trig AB Qs

1.(e) $f(x) = (1-x)\sin^{-1}(\sqrt{x})$

$$u = (1-x) \quad v = \sin^{-1}(\sqrt{x})$$

$$u' = -1 \quad v' = \frac{1}{\sqrt{1-(\sqrt{x})^2}} \cdot \frac{1}{2}x^{-1/2}$$

$$= \frac{1}{2\sqrt{x}\sqrt{1-x}}$$

$$f'(x) = -\sin^{-1}(\sqrt{x}) + (1-x) \cdot \frac{1}{2\sqrt{x}\sqrt{1-x}}$$

$$= -\sin^{-1}(\sqrt{x}) + \frac{(\sqrt{1-x}\,\cancel{\sqrt{1-x}})}{2\sqrt{x}\,\cancel{\sqrt{1-x}}}$$

$$= \frac{\sqrt{1-x}}{2\sqrt{x}} - \sin^{-1}(\sqrt{x})$$

1(f) $f(x) = (4+x^2)\tan^{-1}\left(\frac{x}{2}\right)$

$$u = (4+x^2) \quad v = \tan^{-1}\left(\frac{x}{2}\right)$$

$$u' = 2x \quad v' = \frac{1}{1+\left(\frac{x}{2}\right)^2} \cdot \frac{1}{2}$$

$$= \frac{1}{\left(1+\frac{x^2}{4}\right)} \cdot \frac{1}{2}$$

$$= \frac{1}{2\left(\frac{4+x^2}{4}\right)}$$

$$= \frac{2}{(4+x^2)}$$

$$f'(x) = u'v + uv'$$

$$= 2x\tan^{-1}\left(\frac{x}{2}\right) + (4+x^2) \cdot \frac{2}{(4+x^2)}$$

$$= 2x\tan^{-1}\left(\frac{x}{2}\right) + 2$$

$$= 2\left(x\tan^{-1}\left(\frac{x}{2}\right) + 1\right)$$

Further Diff → IMPLICIT. A/B

Q2. a) $ye^x + xe^y = 1$

(ye^x) ②

① $u = y \quad v = e^x$

$u' = \frac{dy}{dx} \quad v' = e^x$

$\left(ye^x + e^x\frac{dy}{dx}\right) + \left(e^y + xe^y\frac{dy}{dx}\right) = 0$

(xe^y)

② $u = x \quad v = e^y$

$u' = 1 \quad v' = e^y\frac{dy}{dx}$

$(e^x + xe^y)\frac{dy}{dx} = -ye^x - e^y$

$$\frac{dy}{dx} = \frac{-ye^x - e^y}{(e^x + xe^y)}$$

Q2 (b) $\ln y = x^2y, \ y \geq 1$

$u = x^2 \quad v = y$

$u' = 2x \quad v' = \frac{dy}{dx}$

$\frac{1}{y}\frac{dy}{dx} = 2xy + x^2\frac{dy}{dx}$

$\frac{1}{y}\frac{dy}{dx} - x^2\frac{dy}{dx} = 2xy$

$\left(\frac{1}{y} - x^2\right)\frac{dy}{dx} = 2xy$

$$\frac{dy}{dx} = \frac{2xy}{\left(\frac{1}{y} - x^2\right)} = \frac{2xy}{\left(\frac{1-x^2y}{y}\right)} = \frac{2xy^2}{1-x^2y}$$

Not well presented, try to get common denominator so express as $\frac{a}{b}$

Q3. $\dfrac{y}{(1+y)} + \dfrac{x}{(1+x)} - x^2y^4 = 0$ at $(1,1)$

---

① $\dfrac{y}{1+y}$ $\boxed{\begin{array}{ll} u=y & v=1+y \\ u'=\dfrac{dy}{dx} & v'=\dfrac{dy}{dx} \end{array}}$

$$\frac{(1+y)\frac{dy}{dx} - y\frac{dy}{dx}}{(1+y)^2} = \frac{\frac{dy}{dx} + \cancel{y\frac{dy}{dx}} - \cancel{y\frac{dy}{dx}}}{(1+y)^2}$$

$$= \frac{1}{(1+y)^2}\cdot\frac{dy}{dx}$$

---

② $\dfrac{x}{(1+x)}$ $\boxed{\begin{array}{ll} u=x & v=1+x \\ u'=1 & v'=1 \end{array}}$ $\dfrac{(1+x)-x}{(1+x)^2} = \dfrac{1}{(1+x)^2}$

---

③ $-x^2y^4$ $\boxed{\begin{array}{ll} u=-x^2 & v=y^4 \\ u'=-2x & v'=4y^3\dfrac{dy}{dx} \end{array}}$ $-2xy^4 - 4x^2y^3\dfrac{dy}{dx}$

$$\frac{1}{(1+y)^2}\cdot\frac{dy}{dx} + \frac{1}{(1+x)^2} - 2xy^4 - 4x^2y^3\frac{dy}{dx} = 0$$

$$\left(\frac{1}{(1+y)^2} - 4x^2y^3\right)\frac{dy}{dx} = 2xy^4 - \frac{1}{(1+x)^2}$$

$$\left(\frac{1-4x^2y^3(1+y)^2}{(1+y)^2}\right)\frac{dy}{dx} = \left(\frac{2xy^4(1+x)^2 - 1}{(1+x)^2}\right)$$

$$\therefore \frac{dy}{dx} = \frac{(2xy^4(1+x)^2-1)(1+y)^2}{(1-4x^2y^3(1+y)^2)(1+x)^2}$$

At $(1,1)$

$$m = \frac{(2\times1\times1(1+1)^2-1)(1+1)^2}{(1-4\times(1+1)^2)(1+1)^2} = \frac{(2(4)-1)\cancel{(4)}}{(1-4(4))\cancel{(4)}} = \frac{7}{-15}$$

Eqn: $(y-1) = \frac{7}{-15}(x-1)$

$-15y+15 = 7x-7 \Longrightarrow 7x+15y-22=0$

Q4a) $y^2 = 4x^2 + 1$

$$2y\frac{dy}{dx} = 8x$$

$$\frac{dy}{dx} = \frac{8x}{2y}$$

$$\therefore \frac{dy}{dx} = \frac{4x}{y}$$

$$\frac{d^2y}{dx^2} = \frac{u'v - uv'}{v^2}$$

$$\boxed{\begin{array}{ll} u = 4x & v = y \\ u' = 4 & v' = \frac{dy}{dx} \end{array}}$$

$$\frac{d^2y}{dx^2} = \frac{4y - 4x\left(\frac{dy}{dx}\right)}{y^2}$$

$$= \frac{4y - 4x\left(\frac{4x}{y}\right)}{y^2}$$

$$= \frac{\left(\frac{4y^2 - 16x^2}{y}\right)}{y^2}$$

$$\therefore \frac{d^2y}{dx^2} = \frac{4y^2 - 16x^2}{y^3}$$

Q4(b) $2y^2 - x^2 = 3y + 3$

$$4y\frac{dy}{dx} - 2x = 3\frac{dy}{dx}$$

$$(4y - 3)\frac{dy}{dx} = 2x$$

$$\frac{dy}{dx} = \frac{2x}{(4y-3)}$$

$$\boxed{\begin{array}{ll} u = 2x & v = 4y - 3 \\ u' = 2 & v' = 4\frac{dy}{dx} \end{array}}$$

$$\frac{d^2y}{dx^2} = \frac{2(4y-3) - 8x\frac{dy}{dx}}{(4y-3)^2}$$

$$= \cancel{8y - 6 - 8x}$$

$$\frac{2(4y-3) - 8x\left(\frac{2x}{4y-3}\right)}{(4y-3)^2}$$

$$= \frac{\left(\frac{2(4y-3)^2 - 16x^2}{(4y-3)}\right)}{(4y-3)^2}$$

$$= \frac{2(4y-3)^2 - 16x^2}{(4y-3)^3}$$

$$= \frac{2\left[(4y-3)^2 - 8x^2\right]}{(4y-3)^3}$$

Implicit A/B Style

Q5.

$$x^2 - y^2 = 1$$

$$2x - 2y\frac{dy}{dx} = 0$$

$$-2y\frac{dy}{dx} = -2x$$

$$\frac{dy}{dx} = \frac{x}{y}$$

$$\boxed{\begin{matrix} u = x & v = y \\ u' = 1 & v' = \dfrac{dy}{dx} \end{matrix}}$$

$$\frac{d^2y}{dx^2} = \frac{y - x\frac{dy}{dx}}{y^2}$$

$$= \frac{y - x\left(\frac{x}{y}\right)}{y^2}$$

$$= \frac{y - \frac{x^2}{y}}{y^2}$$

$$= \frac{\left(\frac{y^2 - x^2}{y}\right)}{y^2}$$

$$\therefore \frac{d^2y}{dx^2} = \frac{y^2 - x^2}{y^3}$$

$$y\left(\frac{d^2y}{dx^2}\right) + \left(\frac{dy}{dx}\right)^2 = y\left(\frac{y^2 - x^2}{y^3}\right) + \left(\frac{x}{y}\right)^2$$

$$= \frac{y^2 - x^2}{y^2} + \frac{x^2}{y^2}$$

$$= \frac{y^2 - x^2 + x^2}{y^2}$$

$$= \frac{y^2}{y^2} = 1 \quad \text{As Required}$$