66

$$\boxed{f'(x) = \frac{f(x+h) - f(x)}{h}}$$

Q1. (a) $f(x) = 2x^2$

$$f'(x) = \frac{2(x+h)^2 - 2x^2}{h}$$

$$= \frac{2(x^2+2xh+h^2) - 2x^2}{h}$$

$$= \frac{\cancel{2x^2} + 4xh + 2h^2 - \cancel{2x^2}}{h}$$

$$= \frac{4x\cancel{h} + 2h^{\cancel{2}}}{\cancel{h}}$$

$$= 4x + 2\cancel{h}$$

As $h \to 0$ lim $f(x) = \underline{\underline{4x}}$

3

(b) $f(x) = 2x^{-2}$ or $\frac{2}{x^2}$

$$f'(x) = \frac{\frac{2}{(x+h)^2} - \frac{2}{x^2}}{h}$$

$$= \frac{\frac{2x^2 - 2(x+h)^2}{x^2(x+h)^2}}{h}$$

$$= \frac{2x^2 - 2(x^2+2xh+h^2)}{x^2h(x+h)^2}$$

$$= \frac{\cancel{2x^2} - \cancel{2x^2} - 4xh - 2h^2}{x^2h(x+h)^2}$$

lim As $h \to 0$ $= \frac{-4x}{x^2(x)^2} = \frac{-4x}{x^4}$

$$f'(x) = \underline{\underline{\frac{-4}{x^3}}}$$

4

(c) $f(x) = e^x$

$$f'(x) = \frac{e^{x+h} - e^x}{h}$$

$$= \frac{e^x \times e^h - e^x}{h}$$

$$= \frac{e^x (e^h - 1)}{h}$$

As $h \to 0$

$$\lim f'(x) = \lim_{h\to 0}(e^x) \times \lim_{h\to 0}\left(\frac{e^h - 1}{h}\right)$$

$$= e^x \times \left(\frac{e^{0.000001} - 1}{0.000001}\right)$$

$$= e^x \times 1$$

$$f'(x) = e^x$$

Check on calculator and you will see $\frac{e^h - 1}{h} \to 1$ as $h \to 0$

4

AH Homework LO2 U1

Q 2. (a) $f(x) = (x^2+3x+5)^{-2}$

$f'(x) = -2(x^2+3x+5)^{-3} . (2x+3)$

$= \dfrac{-2(2x+3)}{(x^2+3x+5)^3}$

2

(b) $f(x) = \tan^3 x$

$= (\tan x)^3$

$f'(x) = 3(\tan x)^2 . \sec^2 x$

$= 3 \tan^2 x \sec^2 x$

2

(c) $f(x) = \sin^2(2x - \frac{\pi}{6})$

$f(x) = [\sin(2x - \frac{\pi}{6})]^2$

$f'(x) = 2[\sin(2x - \frac{\pi}{6})]^1 . \cos(2x - \frac{\pi}{6}) . 2$

$= 4 \sin(2x - \frac{\pi}{6}) \cos(2x - \frac{\pi}{6})$

3

(d) $f(x) = \sqrt{x} \ln x$

Let $u = x^{1/2}$ $\quad v = \ln x$

$u' = \frac{1}{2} x^{-1/2}$ $\quad v' = \frac{1}{x}$

$= \dfrac{1}{2\sqrt{x}}$

$f'(x) = u'v + uv'$

$= \dfrac{1}{2\sqrt{x}} \times \ln x + \sqrt{x} \times \dfrac{1}{x}$

$= \dfrac{1}{2\sqrt{x}} \ln x + \dfrac{\cancel{\sqrt{x}}}{\sqrt{x}\cancel{\sqrt{x}}}$

$= \dfrac{1}{2\sqrt{x}} (\ln x + 2)$

3

(e) $f(x) = x^4 e^{3x}$

Let $u = x^4$ $\quad v = e^{3x}$

$u' = 4x^3$ $\quad v' = 3e^{3x}$

$f'(x) = u'v + uv'$

$= 4x^3 e^{3x} + 3x^4 e^{3x}$

$= x^3 e^{3x} (4 + 3x)$

3

LO2 U1 AH Homework

Q2. (f) $f(x) = \dfrac{x^2}{2x+3}$

$$\boxed{\begin{array}{ll} u = x^2 & v = 2x+3 \\ u' = 2x & v' = 2 \end{array}}$$

$$f'(x) = \frac{u'v - uv'}{v^2}$$

$$= \frac{2x(2x+3) - 2x^2}{(2x+3)^2}$$

$$= \frac{4x^2 + 6x - 2x^2}{(2x+3)^2}$$

$$= \frac{2x^2 + 6x}{(2x+3)^2}$$

$$= \frac{2x(x+3)}{(2x+3)^2}$$

3

(g) $f(x) = \dfrac{x^2 \ln x}{x+1}$

$$\boxed{\begin{array}{ll} a = x^2 & b = \ln x \\ a' = 2x & b' = \frac{1}{x} \end{array}}$$

for Top $x^2 \ln x$

$$\frac{d(\text{Top})}{dx} = 2x\ln x + x$$

$u = x^2 \ln x \qquad v = x+1$

$u' = x(2\ln x + 1) \qquad v' = 1.$

$$f'(x) = \frac{x(x+1)(2\ln x + 1) - x^2 \ln x}{(x+1)^2}$$

$$= \frac{2x^2 \ln x + x^2 + 2x\ln x + x - x^2 \ln x}{(x+1)^2}$$

$$= \frac{x^2 \ln x + 2x\ln x + x + x^2}{(x+1)^2}$$

$$= \frac{x(x\ln x + 2\ln x + x + 1)}{(x+1)^2}$$

4

Q2 (h) $f(x) = x^2 e^{\cos x}$

$u = x^2 \quad v = e^{\cos x}$

$u' = 2x \quad v' = -\sin x\, e^{\cos x}$

$f'(x) = u'v + uv'$

$= 2x e^{\cos x} - x^2 \sin x\, e^{\cos x}$

$= e^{\cos x}(2x - x^2 \sin x)$

$= x e^{\cos x}(2 - x\sin x)$

3

Q2 (i) $f(x) = \dfrac{\sin x}{x^2}$

$u = \sin x \quad v = x^2$

$u' = \cos x \quad v' = 2x$

$f'(x) = \dfrac{u'v - uv'}{v^2} = \dfrac{x^2\cos x - 2x\sin x}{x^4}$

$= \dfrac{x(x\cos x - 2\sin x)}{x^4}$

$= \dfrac{x\cos x - 2\sin x}{x^3}$

$\left(\text{OR} \quad \dfrac{\cos x}{x^2} - \dfrac{2\sin x}{x^3}\right)$

3

Q3. (a) $\sec x = \dfrac{1}{\cos x}$; $\operatorname{cosec} x = \dfrac{1}{\sin x}$; $\cot x = \dfrac{1}{\tan x}$

3

(b) $f(x) = \dfrac{1}{\cos x}$

$\begin{pmatrix} u = 1 & v = \cos x \\ u' = 0 & v' = -\sin x \end{pmatrix}$

$f'(x) = \dfrac{0 + \sin x}{\cos^2 x}$

$= \dfrac{\sin x}{\cos x} \times \dfrac{1}{\cos x}$

$= \dfrac{\tan x}{\cos x}$

$= \tan x \sec x$

$\therefore f(x) = \sec x$

$f'(x) = \tan x \sec x$

$f(x) = \dfrac{1}{\sin x} \quad \begin{pmatrix} u = 1 & u' = 0 \\ v = \sin & v' = \cos \end{pmatrix}$

$f'(x) = \dfrac{0 - \cos x}{\sin^2 x}$

$= -\dfrac{\cos x \times 1}{\sin x \times \sin x}$

$= -\dfrac{1}{\tan x} \times \operatorname{cosec} x$

$= -\cot x \operatorname{cosec} x$

$\therefore f(x) = \operatorname{cosec} x$

$f'(x) = -\cot x \operatorname{cosec} x$

$f(x) = \dfrac{\cos x}{\sin x}$ $\quad u = c,\ u' = -s,\ v = s,\ v' = c$

$f'(x) = \dfrac{-\sin^2 x - \cos^2 x}{\sin^2 x}$

$= \dfrac{-(\sin^2 x + \cos^2 x)}{\sin^2 x}$

$= \dfrac{-1}{\sin^2 x}$

$= -\operatorname{cosec}^2 x$

$\therefore f(x) = \cot x$

$f'(x) = -\operatorname{cosec}^2 x$

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LO2 U1 AH Homework.

Q3. (c) Recall: $f(x) = \sec x$ $\quad f'(x) = \tan x \sec x$ $\quad f(x) = \operatorname{cosec} x$ $\quad f'(x) = -\cot x \operatorname{cosec} x$

$f(x) = \cot x$

$f'(x) = -\operatorname{cosec}^2 x$

(i) $f(x) = \cot 3x$ $\qquad$ If $\sin 3x \to 3\cos 3x$

$f'(x) = -3\operatorname{cosec}^2 3x$

2

(ii) $f(x) = \operatorname{cosec}^2 x = (\operatorname{cosec} x)^2$

3

$f'(x) = 2(\operatorname{cosec} x)^1 \cdot -\cot x \operatorname{cosec} x$

$= -2\cot x \operatorname{cosec}^2 x$ (OR $-2\cos x \operatorname{cosec}^3 x$)

(iii) $f(x) = 2x^2 \sec x$ $\qquad$ $f'(x) = u'v + uv'$

$u = 2x^2 \quad v = \sec x$

$u' = 4x \quad v' = \tan x \sec x$

$f'(x) = 4x \sec x + 2x^2 \tan x \sec x$

$= 2x \sec x \,(2 + x\tan x)$

3

Q4.

$$\boxed{y = \frac{\sin x}{x^2}} \qquad \frac{dy}{dx} = \frac{x^2\cos x - 2x\sin x}{x^4}$$

$u = \sin x \qquad v = x^2$

$u' = \cos x \qquad v' = 2x$

$$= \underline{\underline{\frac{\cos x}{x^2} - \frac{2\sin x}{x^3}}}$$

$$\frac{d^2y}{dx^2} = \left[\left(\frac{a'b - ab'}{b^2}\right) - \left(\frac{c'd - cd'}{d^2}\right)\right]$$

$$\boxed{\begin{matrix} a = \cos x & b = x^2 \\ a' = -\sin x & b' = 2x \end{matrix}} \quad \boxed{\begin{matrix} c = 2\sin x & d = x^3 \\ c' = 2\cos x & d' = 3x^2 \end{matrix}}$$

$$d^2y/dx^2 = \left(\frac{-x^2\sin x - 2x\cos x}{x^4}\right) - \left(\frac{2x^3\cos x - 6x^2\sin x}{x^6}\right)$$

$$= -\frac{\sin x}{x^2} - \frac{2\cos x}{x^3} - \frac{2\cos x}{x^3} + \frac{6\sin x}{x^4}$$

$$\frac{d^2y}{dx^2} = \underline{\underline{-\frac{\sin x}{x^2} - \frac{4\cos x}{x^3} + \frac{6\sin x}{x^4}}}$$

Then,

$$x^2\frac{d^2y}{dx^2} + 4x\frac{dy}{dx} + (x^2+2)y$$

$$= x^2\left[-\frac{\sin x}{x^2} - \frac{4\cos x}{x^3} + \frac{6\sin x}{x^4}\right] + 4x\left[\frac{\cos x}{x^2} - \frac{2\sin x}{x^3}\right] + (x^2+2)\left(\frac{\sin x}{x^2}\right)$$

$$= \cancel{-\sin x} - \cancel{\frac{4\cos x}{x}} + \cancel{\frac{6\sin x}{x^2}} + \cancel{\frac{4\cos x}{x}} - \cancel{\frac{8\sin x}{x^2}} \cancel{+ \sin x} + \cancel{\frac{2\sin x}{x^2}}$$

$= \underline{\underline{0}}$ as required.

9

Q4 Alternative

$y = \frac{\sin x}{x^2}$

$u = \sin x \quad v = x^2$
$u' = \cos x \quad v' = 2x$

(a) $\frac{dy}{dx} = \frac{u'v - uv'}{v^2} = \frac{x^2\cos x - 2x\sin x}{(x^2)^2} = \frac{x\cos x - 2\sin x}{x^3}$ (3)

(b) $\frac{d^2y}{dx} = \frac{u'v - uv'}{v^2}$

But $x\cos x$ is product nested within the quotient rule

Consider $x\cos x$

Let $a = x \quad b = \cos x$
$a' = 1 \quad b' = -\sin x$

$f'(x) = a'b + ab' = \cos x - x\sin x$

$\frac{dy}{dx} = \frac{x\cos x - 2\sin x}{x^3}$

$u = (x\cos x) - 2\sin x$
$u' = (\cos x - x\sin x) - 2\cos x$
$= -\cos x - x\sin x$
$v = x^3$
$v' = 3x^2$

$\frac{d^2y}{dx^2} = \frac{u'v - uv'}{v^2}$

$= \frac{(-\cos x - x\sin x)x^3 - 3x^2(x\cos x - 2\sin x)}{(x^3)^2}$

$= \frac{-x^3\cos x - x^4\sin x - 3x^3\cos x + 6x^2\sin x}{x^6}$

$= \frac{-4x^3\cos x - x^4\sin x + 6x^2\sin x}{x^6}$ (3)

$= \frac{6\sin x - x^2\sin x - 4x\cos x}{x^4}$

Recap. $y = \frac{\sin x}{x^2}$; $\frac{dy}{dx} = \frac{x\cos x - 2\sin x}{x^3}$

& $\frac{d^2y}{dx^2} = \frac{6\sin x - x^2\sin x - 4x\cos x}{x^4}$

$$x^2 \frac{d^2y}{dx^2} + 4x\frac{dy}{dx} + (x^2+2)y$$

$$= x^2\left[\frac{6\sin x - x^2\sin x - 4x\cos x}{x^4}\right] + 4x\left[\frac{x\cos x - 2\sin x}{x^3}\right]$$

$$+ \left(\frac{x^2+2}{1}\right)\left(\frac{\sin x}{x^2}\right)$$

$$= \frac{6\sin x - x^2\sin x - 4x\cos x}{x^2} + \frac{4x\cos x - 8\sin x}{x^2}$$

$$+ \frac{x^2\sin x + 2\sin x}{x^2}$$

$$= \frac{\cancel{6\sin x} - \cancel{x^2\sin x} - \cancel{4x\cos x} + \cancel{4x\cos x} - \cancel{8\sin x} + \cancel{x^2\sin x} + \cancel{2\sin x}}{x^2}$$

$$= \frac{0}{x^2}$$

3

$= 0$ As required.