Q1. a)

$$f(x) = 6\tan^{-1}\sqrt{x}$$

$$= 6 \cdot \left(\frac{1}{1+(\sqrt{x})^2}\right) \cdot \frac{1}{2\sqrt{x}}$$

$u = \sqrt{x} = x^{1/2}$

$u' = \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}}$

Remember the CHAIN RULE

$$= \frac{6}{2\sqrt{x}(1+x)}$$

$$= \frac{3}{\sqrt{x}(1+x)}$$

3

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b)

$$y = x^{x-2}$$

When exponential problem to simplify take 'ln' of both sides

$$\ln|y| = \ln|x^{x-2}|$$

$$\ln y = (x-2)\ln|x|$$

$$\frac{1}{y}\cdot\frac{dy}{dx} = \ln|x| + \frac{(x-2)}{x}$$

Product Rule

$u = (x-2) \quad v = \ln|x|$

$u' = 1 \quad v' = 1/x$

$u'v + uv' = \ln|x| + \left(\frac{x-2}{x}\right)$

$$\frac{dy}{dx} = \left(\frac{x\ln|x| + (x-2)}{x}\right) \times y$$

(*)

$$= \left(\frac{x\ln|x| + (x-2)}{x}\right) \times x^{x-2}$$

3

Can be further simplified to $(x\ln|x| + (x-2)) \times x^{x-3}$

(*) If don't find common denominator shall accept

$$\left(\ln|x| + \frac{(x-2)}{x}\right)x^{x-2} \quad \text{OR} \quad \left(\ln|x| + 1 - \frac{2}{x}\right)x^{x-2}$$

Q2. $z_1 = 2i$ & $z_2 = 1-i$

a) $$\frac{z_1}{z_2} = \frac{2i}{1-i} = \frac{2i}{1-i} \times \frac{(1+i)}{(1+i)} = \frac{2i(1+i)}{1-i^2} = \frac{2i+2i^2}{1-(-1)} = \frac{-2+2i}{2}$$

$$\therefore \frac{z_1}{z_2} = -1+i$$

Need conjugate to simplify denominator

2

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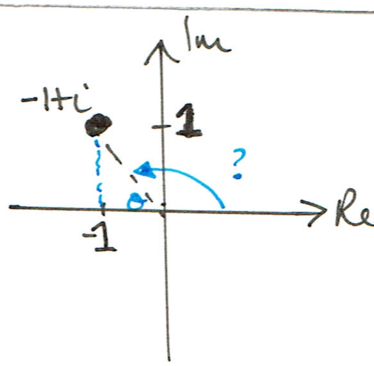
b) $$\arg\left(\frac{z_1}{z_2}\right)$$



Need to find Quadrant in :-

Recall

$z = -1+i$
$= a+bi \Rightarrow a=-1$ & $b=1$

First quadrant: $$\arg(z) = \tan^{-1}|b/a| = \tan^{-1}|1/{-1}| = \tan^{-1}(1) = \pi/4$$

As in 2nd Quadrant

1

$$\arg\left(\frac{z_1}{z_2}\right) = \pi - \alpha = \pi - \frac{\pi}{4} = \frac{3\pi}{4}$$

Q3. $\left(3p^3 - \frac{2}{p}\right)^4 = \sum_{r=0}^{4} \binom{4}{r} (3p^3)^{4-r} \left(-\frac{2}{p}\right)^r$

$$\boxed{\text{Recall } (x+y)^n = \sum_{r=0}^{n} \binom{n}{r} (x)^{n-r} (y)^r}$$

$$\binom{4}{r} (3p^3)^{4-r} \left(\frac{-2}{p}\right)^r$$

$$\boxed{\binom{4}{r} (3)^{4-r} (-2)^r} \times \boxed{(p^3)^{4-r} \left(\frac{1}{p}\right)^r}$$

Split to find co-eff & term separately.

Coeff: $\binom{4}{r} (3)^{4-r} (-2)^r$

Term: $(p^3)^{4-r} (p^{-1})^r = p^0$

$p^{12-3r} \times p^{-r} = p^0$

$p^{12-4r} = p^0$

$12 - 4r = 0$

$-4r = -12$

$\therefore \underline{\underline{r = 3}}$

$C = \binom{4}{3} (3)^{4-3} (-2)^3$

$= \frac{4!}{3!\,1!} \times (3)^1 \times (-2)^3$

$= \frac{4 \times 3!}{3! \times 1!} \times 3 \times -8$

$= \underline{\underline{-96}}$

4

Q4

$$\begin{aligned} a + 4b - c &= 0 \\ -2a - 7b \quad &= 3 \\ a + 6b - 5c &= 2 \end{aligned} \rightarrow \begin{array}{c} \left(\begin{array}{ccc|c} 1 & 4 & -1 & 0 \\ -2 & -7 & 0 & 3 \\ 1 & 6 & -5 & 2 \end{array}\right) \end{array} \begin{array}{c} r_1 \\ r_2 \\ r_3 \end{array}$$

$$\begin{array}{c} 2r_1 \\ r_2 \\ 2r_3 \end{array} \left(\begin{array}{ccc|c} 2 & 8 & -2 & 0 \\ -2 & -7 & 0 & 3 \\ 2 & 12 & -10 & 4 \end{array}\right) \begin{array}{c} r_1 \\ r_2 \\ r_3 \end{array} \rightarrow \begin{array}{c} r_1 \\ r_1 + r_2 \\ r_3 - r_1 \end{array} \left(\begin{array}{ccc|c} 2 & 8 & -2 & 0 \\ 0 & 1 & -2 & 3 \\ 0 & 4 & -8 & 4 \end{array}\right) \begin{array}{c} r_1 \\ r_2 \\ r_3 \end{array}$$

$$\begin{array}{c} \frac{1}{2}r_1 \\ 1 r_2 \\ \frac{1}{4}r_3 \end{array} \left(\begin{array}{ccc|c} 1 & 4 & -1 & 0 \\ 0 & 1 & -2 & 3 \\ 0 & 1 & -2 & 1 \end{array}\right) \begin{array}{c} r_1 \\ r_2 \\ r_3 \end{array} \rightarrow \begin{array}{c} r_1 \\ r_2 \\ r_3 - r_2 \end{array} \left(\begin{array}{ccc|c} 1 & 4 & -1 & 0 \\ 0 & 1 & -2 & 3 \\ 0 & 0 & 0 & -2 \end{array}\right) \begin{array}{c} r_1 \\ r_2 \\ r_3 \end{array}$$

As $0x + 0y + 0z = -2 \Rightarrow$ Impossible!

5

$\therefore$ No Solutions exist as INCONSISTENT

Q5.

If $2^{3n}-1$ is divisible by 7

Then $2^{3n}-1 = 7m$, where m is a constant/integer

Let n=1 $2^{3(1)}-1 = 8-1 = 7 = 7\times1 \Rightarrow$ true for n=1

Assume true for n=k:

$$2^{3k}-1 = 7m$$

Consider for n=k+1:

$$2^{3(k+1)}-1 = 2^{3k+3}-1$$

$$= 2^{3k}.2^{3}-1$$ (Separate using indice rules.)

$$= 2^{3k}\times 2^{3}-1$$

$$= 8\times 2^{3k}-1$$ (Note must manipulate to create CF of 8)

$$= 8\times 2^{3k}-1+7-7$$

$$= 8\times 2^{3k}-8+7$$

$$= 8(2^{3k}-1)+7$$ (Using Assumption when n=k)

$$= 8(7m)+7$$

$$= 7(8m+1)$$

$\boxed{\text{Let } N=(8m+1)}$ $\rightarrow$ $= 7N$

5

Thus as true for n=1, assumed true for n=k and by Proof by Induction also true for n=k+1, must be true for all $n\in N$. Thus $2^{3n}-1$ is divisible by 7 for all n.

Q6.

$$|z-3i| = |z+2|$$

$$|(x+iy)-3i| = |(x+iy)+2|$$

$$|(x)+i(y-3)| = |(x+2)+i(y)|$$

$$\sqrt{x^2+(y-3)^2} = \sqrt{(x+2)^2+y^2}$$

$$x^2+(y-3)^2 = (x+2)^2+y^2$$

$$\cancel{x^2}+\cancel{y^2}-6y+9 = \cancel{x^2}+4x+4+\cancel{y^2}$$

3

$$-6y+9 = 4x+4$$

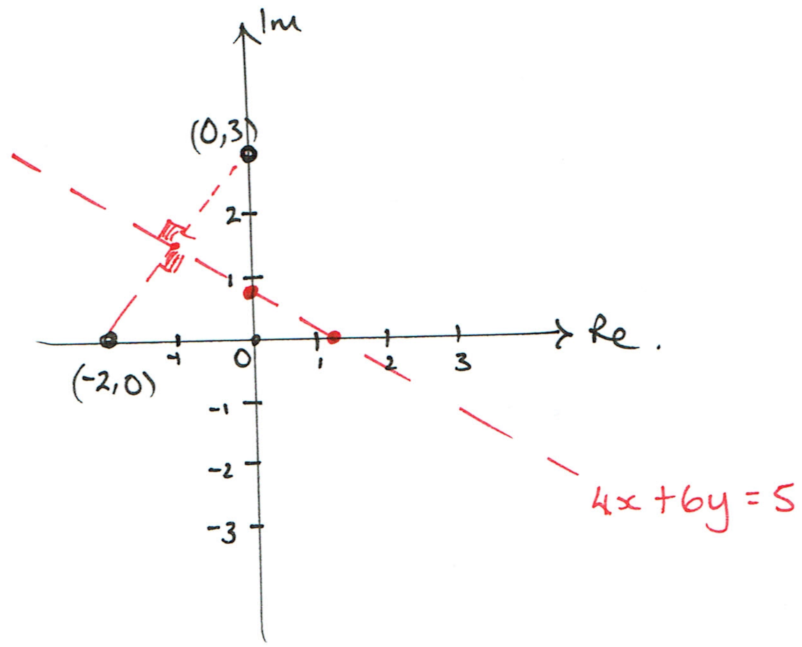
$$4x+6y-5=0$$

$$4x+6y=5$$

OR

$$\boxed{\text{OR} \quad y = -\frac{2}{3}x+\frac{5}{6}}$$

Equation can be given in various formats



Q7. $\int_0^1 2\tan^{-1}x\,dx$

$$\boxed{\int u'v = uv - \int uv'}$$

✱ Must Diff. $\tan^{-1}(x)$

Let $u = \tan^{-1}(x)$ $\quad v' = 2x$

$u' = \frac{1}{1+x^2}$ $\quad v' = 2$

$$\int_0^1 2\tan^{-1}(x)\,dx = \left[2x\tan^{-1}(x)\right]_0^1 - \int_0^1 \frac{2x}{1+x^2}\,dx$$

(for all of line)

$$= \left[2x\tan^{-1}(x)\right]_0^1 - \int_1^2 \frac{\cancel{2x}}{u}\cdot\frac{du}{\cancel{2x}}$$

$$= \left[2x\tan^{-1}(x)\right]_0^1 - \left[\ln|u|\right]_1^2$$

$$= \left(2\tan^{-1}(1) - 0\right) - \left(\ln|2| - \ln|1|\right)$$

$$= 2\times\frac{\pi}{4} - \ln|2|$$

$$= \underline{\underline{\frac{\pi}{2} - \ln|2|}}$$

Substitution

Let $u = 1+x^2$

$\frac{du}{dx} = 2x$

$\therefore \frac{du}{2x} = dx$

$\underline{x=0}: u = 1+0 = \underline{\underline{1}}$

$\underline{x=1}: u = 1+1 = \underline{\underline{2}}$

$\therefore \int_1^2$

4

Q8. $x = t^2 - 2t \qquad y = 1 - t^4$

At $t=-1$: $x = (-1)^2 - 2(-1)$ $\qquad y = 1 - (-1)^4$

$= 1 + 2 \qquad\qquad = 1 - 1$

$\therefore x = 3 \qquad \therefore y = 0 \Rightarrow (3,0)$

$x = t^2 - 2t \qquad y = 1 - t^4$

$\frac{dx}{dt} = 2t - 2 \qquad \frac{dy}{dt} = -4t^3$

$$\therefore \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{-4t^3}{2(t-1)} = \frac{-2t^3}{(t-1)}$$

At $t=-1$ Gradient, $m = \frac{dy}{dx} = \frac{-2t^3}{t-1} = \frac{-2(-1)^3}{(-1)-1} = \frac{-2\times -1}{-2} = -1$

At $m=-1$ & $(3,0)$ Equation of Tangent is:-

$$(y-b) = m(x-a)$$

$$(y-0) = -1(x-3)$$

$$y = -x + 3$$

OR $x + y - 3 = 0$

4

Q9. $f(x) = \dfrac{x^3 + 3x^2 - 8x + 2}{x^2 - 2x + 1} \Rightarrow$

$$\begin{array}{r|l} & x + 5 \\ \hline x^2 - 2x + 1 & x^3 + 3x^2 - 8x + 2 \\ & x^3 - 2x^2 + x \\ \hline & \quad 5x^2 - 9x + 2 \\ & \quad 5x^2 - 10x + 5 \\ \hline & \qquad x - 3 \end{array}$$

$$\therefore f(x) = x + 5 + \frac{x - 3}{x^2 - 2x + 1}$$

$$= x + 5 + \frac{x - 3}{(x-1)^2}$$

$$\frac{x-3}{(x-1)^2} = \frac{A}{(x-1)} + \frac{B}{(x-1)^2}$$

$\therefore A(x-1) + B = x - 3$

Let $x = 1$: $B = 1 - 3 \Rightarrow B = -2$

Let $x = 2$: $A + B = 2 - 3$

$A - 2 = -1$

$\therefore A = 1$

$$= x + 3 + \frac{1}{(x-1)} + \frac{(-2)}{(x-1)^2}$$

$$\therefore f(x) = x + 3 + \frac{1}{(x-1)} - \frac{2}{(x-1)^2}$$

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Q10. $\displaystyle\int_0^{\pi/4} \frac{\sec^2 x}{1+\tan x}\,dx$

$$= \int_1^2 \frac{\cancel{\sec^2 x}}{t} \cdot \frac{dt}{\cancel{\sec^2 x}}$$

$$= \int_1^2 \frac{dt}{t}$$

$$= \Big[\ln|t|\Big]_1^2$$

$$= \ln|2| - \ln|1|$$

$$= \ln|2|$$

Let $t = 1+\tan x$

$$\frac{dt}{dx} = \sec^2 x$$

$$\frac{dt}{\sec^2 x} = dx$$

If $x=0$: $t = 1+\tan 0 = 1$

If $x = \frac{\pi}{4}$: $t = 1+\tan\frac{\pi}{4} = 1+1 = 2$

5

Q11. a) $12, 15, 18, \ldots, 99$

If $a = 12$; $d = 3$ then $U_n = a + (n-1)d$

$$= 12 + (n-1) \times 3$$
$$= 3n - 3 + 12$$
$$\therefore U_n = 3n + 9$$

Need to find a rule to figure out how many terms you need to find the sum of if the last term is 99

As last term is 99 $\qquad U_n = 3n + 9$

then $3n + 9 = 99$

$$3n = 90$$
$$n = 30$$

4

So finding sum of 1st 30 terms

$$S_n = \frac{n}{2}\left[2a + (n-1)d\right]$$

$$S_{30} = \frac{30}{2}\left[2 \times 12 + (30-1) \times 3\right]$$

$$= 15\left[24 + 29 \times 3\right]$$

$$= 15\left[24 + 87\right]$$

$$= 15 \times 111$$

$$\therefore S_{30} = 1665$$

Q11(b) $1 + \sin^2\theta + \sin^4\theta + \sin^6\theta + \ldots = 2$

$\boxed{a = 1;\ r = \sin^2\theta;\ S_\infty = 2 \ \circledast \text{(Note as NO END TERM} \Rightarrow S_\infty)}$

$S_\infty = \dfrac{a}{1-r}$ & $S_\infty = 2$

Then $\dfrac{a}{1-r} = 2$

$\dfrac{1}{1-\sin^2\theta} = 2$

$\boxed{\begin{array}{l}\sin^2\theta + \cos^2\theta = 1 \\ \therefore \cos^2\theta = 1 - \sin^2\theta\end{array}}$

$\dfrac{1}{(\cos^2\theta)} = 2$

$1 = 2\cos^2\theta$

$\therefore \cos^2\theta = 1/2$ (Must have $\pm$ for Mark)

5

$\cos\theta = \pm\dfrac{1}{\sqrt{2}}$

As $0 < \theta < \frac{\pi}{2}$ $\cos\theta = \dfrac{1}{\sqrt{2}}$

$\boxed{\text{As can only be in first Quadrant}}$

$\therefore \theta = \dfrac{\pi}{4}$

Q12. $\dfrac{dy}{dx} = e^{x+y}$

$\therefore \dfrac{dy}{dx} = e^x \cdot e^y$

Separate Variables

$\dfrac{dy}{e^y} = e^x\,dx$

$\int e^{-y}\,dy = \int e^x\,dx$

Integrate both sides

$\dfrac{e^{-y}}{-1} = e^x + c$

$\dfrac{-1}{e^y} = e^x + c$

Rearrange:

$\therefore \dfrac{-1}{(e^x+c)} = e^y$

If $e^y = \dfrac{-1}{(e^x+c)}$

Find y

$\ln|e^y| = \ln\left|\dfrac{-1}{(e^x+c)}\right|$

$\therefore y = \ln\left|\dfrac{-1}{(e^x+c)}\right|$

Is General Solution

$y=0;\ x=1$

$y = \ln\left|\dfrac{-1}{(e^x+c)}\right|$

$0 = \ln\left|\dfrac{-1}{(e^1+c)}\right|$ Substs Correctly

$\ln|1| = \ln\left|\dfrac{-1}{(e^1+c)}\right|$

$1 = \dfrac{-1}{e+c}$

$e+c = -1$

$\therefore c = (-1-e)$ Finds c

4

If $y = \ln\left|\dfrac{-1}{e^x+c}\right|$ 2

Then

$y = \ln\left|\dfrac{-1}{e^x-1-e}\right|$

So Particular Solution is:

$y = \ln\left|\dfrac{-1}{e^x-e-1}\right|$

[* Need to state c in particular solution for full credit although marking scheme gives mark at c.]

$$f(x) = \frac{x^2+3}{x+1}, \quad x \neq -1, \; x \in R$$

a) (i) Vertical Asymptote : $\underline{x = -1}$ (1)

(When Undefined)

(ii) $f(x) = x - 1 + \frac{4}{(x+1)}$

$$\begin{array}{r|l} & x - 1 \\ \hline x+1 & x^2 \quad +3 \\ & x^2 + x \\ \hline & -x + 3 \\ & -x - 1 \\ \hline & 4 \end{array}$$

$\therefore$ Non-Vertical Asymptote : $y = x - 1$ (2)

(iii) $x = 0$: $y = \frac{0^2+3}{0+1} = 3 \Rightarrow (0, 3)$

$y = 0$: $\frac{x^2+3}{x+1} = 0$

$x^2 + 3 = 0$

$x^2 = -3$

As $x^2 \neq -3$ $\Rightarrow$ Does Not cut x-axis. (2)

(b) $f(x) = x - 1 + 4(x+1)^{-1}$

$f'(x) = 1 - 4(x+1)^{-2} = 1 - \frac{4}{(x+1)^2}$

Stat Pts $f'(x) = 0$: $1 - \frac{4}{(x+1)^2} = 0$

$1 = \frac{4}{(x+1)^2}$

$(x+1)^2 = 4$

$x + 1 = \pm 2$

$x = -1-2 \qquad x = -1+2$

$\underline{x = -3}$ & $\underline{x = 1}$

All 1 Mark

y-coords

$x = 1$: $y = \frac{(-1)^2+3}{(1)+1} = \frac{4}{2} = 2$

$\therefore (1, 2)$

$x = -3$: $y = \frac{(-3)^2+3}{(-3)+1} = \frac{12}{-2}$

$y = -6$

$\therefore (-3, -6)$

$$f(x) = x - 1 + 4(x+1)^{-1}$$

$$f'(x) = 1 - 4(x+1)^{-2}$$

$$f''(x) = 8(x+1)^{-3} = \frac{8}{(x+1)^3}$$

5

At (1,2):

$$f''(x) = \frac{8}{(x+1)^3}$$

$$f''(1) = \frac{8}{(1+1)^3} = 1 > 0 \ \cup$$

MIN

At (-3,-6):

$$f''(x) = \frac{8}{(x+1)^3}$$

$$f''(-3) = \frac{8}{(-3+1)^3} = -1 < 0 \ \cap$$

Max

$\therefore$ Min TPt (1,2) & Max TPt at (-3,-6)

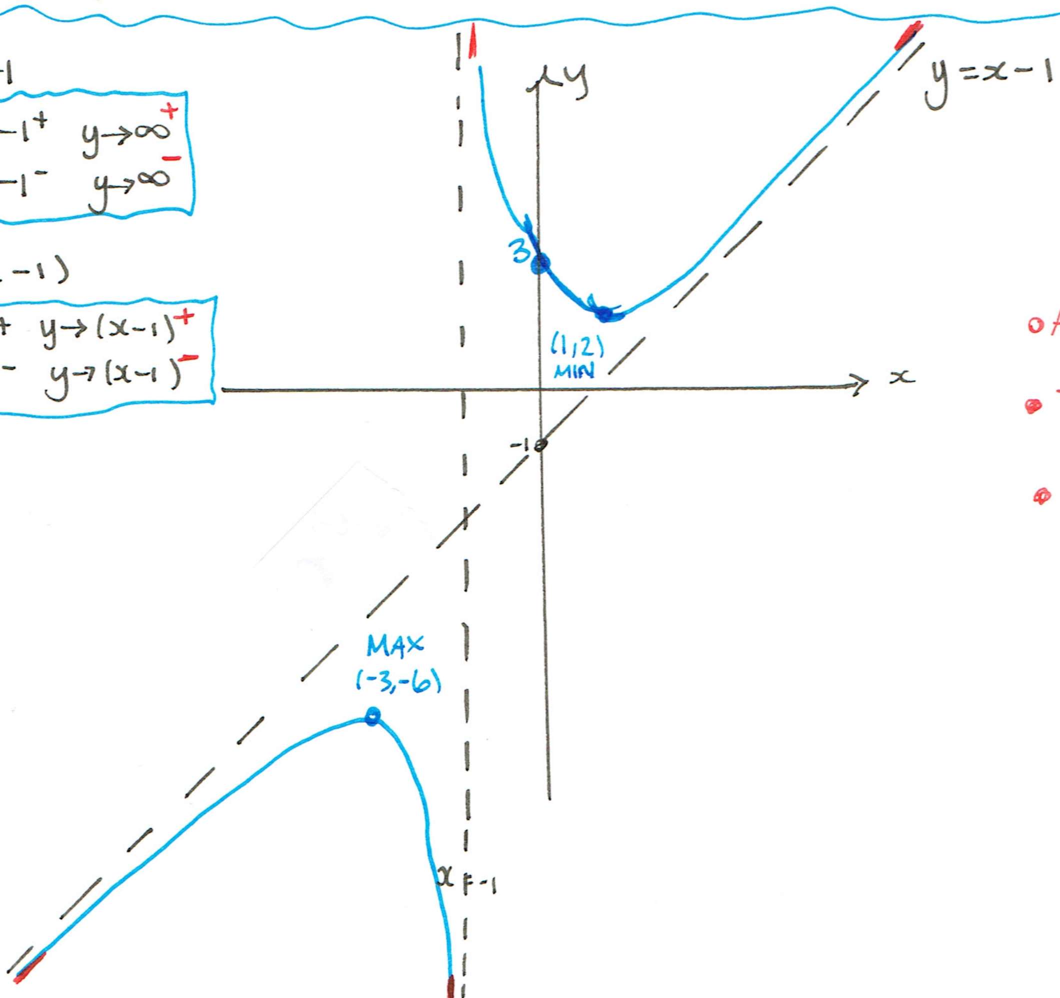
Summary. $x = -1$; $y = x - 1$; (0,3) & TPts

$x = -1$

$x \to -1^+$ $y \to \infty^+$
$x \to -1^-$ $y \to \infty^-$

$y = (x-1)$

$x \to \infty^+$ $y \to (x-1)^+$
$x \to \infty^-$ $y \to (x-1)^-$



- Asymptotes
- Turning Points (Both)
- Sketch + (0,3)

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