Adv 2012/13 Units 1 & 2 Prelim

Q1. $\binom{n}{2} - \binom{n}{1} = 20$

$$\frac{n!}{2!(n-2)!} - \frac{n!}{1!(n-1)!} = 20$$

$$\frac{n(n-1)\cancel{(n-2)!}}{2 \times \cancel{(n-2)!}} - \frac{n\cancel{(n-1)!}}{1 \times \cancel{(n-1)!}} = 20$$

$$\frac{n(n-1)}{2} - n = 20$$

$$n^2 - n - 2n = 40$$

$$n^2 - 3n - 40 = 0$$

$$(n+5)(n-8) = 0$$

$n = -5 \qquad n = 8$

As $n > 0$, $n = 8$

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Q2. $x = 6t \Rightarrow dx/dt = 6$

$y = 2t^3 - 6t^2 \Rightarrow \frac{dy}{dt} = 6t^2 - 12t = 6t(t-2)$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{6t(t-2)}{6} = t(t-2)$$

Stat Pts when $\frac{dy}{dx} = 0$: $t(t-2) = 0$

$t = 0$ & $t = 2$

If $t = 0$

$x = 6(0) = 0$

$y = 2(0)^3 - 6(0)^2 = 0$

$\Rightarrow (0,0)$

If $t = 2$

$x = 6(2) = 12$

$y = 2(2)^3 - 6(2)^2 = 16 - 24 = -8$

$\Rightarrow (12,-8)$

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b) $$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}} = \frac{\frac{d}{dt}(t^2-2t)}{\frac{dx}{dt}} = \frac{2t-2}{6} = \frac{(t-1)}{3}$$

When $t=0$ $\quad \frac{d^2y}{dx^2} = \frac{(0-1)}{3} = \frac{-1}{3} < 0 \Rightarrow \cap$ Max $(0,0)$

When $t=2$ $\quad \frac{d^2y}{dx^2} = \frac{(2-1)}{3} = \frac{1}{3} > 0 \Rightarrow \cup$ Min $(12,-8)$

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$\therefore$ Max TPt $(0,0)$ & Min TPt $(12,-8)$

AH 2012/13 Units 1 & 2 Prelim

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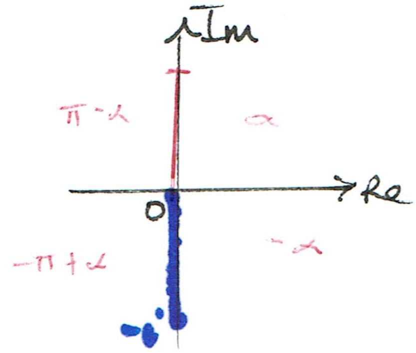
Q3. $z = -i$



$\therefore \ |z| = 1$

In 1st Quad $\alpha = \frac{\pi}{2}$ $\therefore$ As reflected $\arg(z) = -\frac{\pi}{2}$

So $z = -i = 1 \times (\cos(-\frac{\pi}{2}) + i\sin(-\frac{\pi}{2}))$

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then $z^4 = \cos(-\frac{\pi}{2}) + i\sin(-\frac{\pi}{2})$

$\boxed{\text{If to find } 4^{th} \text{ roots} \quad (z)^{1/4} = \left(\cos(-\tfrac{\pi}{2}) + i\sin(-\tfrac{\pi}{2})\right)^{1/4}}$

4 Roots are

$z_1 = \left(\cos\left(\frac{-\pi}{2}\right) + i\sin\left(\frac{-\pi}{2}\right)\right)^{1/4} = \cos\left(\frac{-\pi}{8}\right) + i\sin\left(\frac{-\pi}{8}\right)$

Both

$z_2 = \left(\cos\left(\frac{-\pi}{2} + 2\pi\right) + i\sin\left(\frac{-\pi}{2} + 2\pi\right)\right)^{1/4} = \cos\left(\frac{3\pi}{8}\right) + i\sin\left(\frac{3\pi}{8}\right)$

$z_3 = \left(\cos\left(-\frac{\pi}{2} + 4\pi\right) + i\sin\left(-\frac{\pi}{2} + 4\pi\right)\right)^{1/4} = \cos\left(\frac{7\pi}{8}\right) + i\sin\left(\frac{7\pi}{8}\right)$

Both

$z_4 = \left(\cos\left(\frac{-\pi}{2} + 6\pi\right) + i\sin\left(\frac{-\pi}{2} + 6\pi\right)\right)^{1/4} = \cancel{\cos\left(\frac{11\pi}{8}\right) + i\sin\left(\frac{11\pi}{8}\right)}$ Care

CARE!! Out of Range! $-\pi < \theta < \pi$

So $\cos\left(\frac{11\pi}{8} - 2\pi\right) + i\sin\left(\frac{11\pi}{8} - 2\pi\right)$

$= \cos\left(\frac{-5\pi}{8}\right) + i\sin\left(\frac{-5\pi}{8}\right)$

Q4.

$$\begin{aligned} 3x - y + 4z &= 9 \\ -2x + y - 2z &= -7 \\ 3x - 2y + 2z &= 12 \end{aligned} \rightarrow \begin{array}{c} \left(\begin{array}{ccc|c} 3 & -1 & 4 & 9 \\ -2 & 1 & -2 & -7 \\ 3 & -2 & 2 & 12 \end{array}\right) \end{array} \begin{array}{c} r_1 \\ r_2 \\ r_3 \end{array}$$

$$\begin{array}{c} 2r_1 \\ 3r_2 \\ 2r_3 \end{array} \left(\begin{array}{ccc|c} 6 & -2 & 8 & 18 \\ -6 & 3 & -6 & -21 \\ 6 & -4 & 4 & 24 \end{array}\right) \begin{array}{c} r_1 \\ r_2 \\ r_3 \end{array} \rightarrow \begin{array}{c} r_1 \\ r_1+r_2 \\ r_3-r_1 \end{array} \left(\begin{array}{ccc|c} 6 & -2 & 8 & 18 \\ 0 & 1 & 2 & -3 \\ 0 & -2 & -4 & 6 \end{array}\right) \begin{array}{c} r_1 \\ r_2 \\ r_3 \end{array}$$

$$\begin{array}{c} \frac{1}{2}r_1 \\ r_2 \\ \frac{1}{2}r_3 \end{array} \left(\begin{array}{ccc|c} 3 & -1 & 4 & 9 \\ 0 & 1 & 2 & -3 \\ 0 & -1 & -2 & 3 \end{array}\right) \begin{array}{c} r_1 \\ r_2 \\ r_3 \end{array} \rightarrow \begin{array}{c} r_1 \\ r_2 \\ r_2+r_3 \end{array} \left(\begin{array}{ccc|c} 3 & -1 & 4 & 9 \\ 0 & 1 & 2 & -3 \\ 0 & 0 & 0 & 0 \end{array}\right) \begin{array}{c} r_1 \\ r_2 \\ r_3 \end{array}$$

As $0x + 0y + 0z = 0 \Rightarrow$ Redundant
$\Rightarrow$ Infinite Solutions exist

Let $z = k$ Then $y + 2k = -3$ & $3x - y + 4z = 9$

$y = (-3 - 2k)$ $\quad 3x + 3 + 2k + 4k = 9$

$3x = 6 - 6k$

$x = 2 - 2k$

$\therefore$ Infinite Solutions exist
if $z = k$; $y = -3 - 2k$ & $x = 2 - 2k$

ie $(-2k + 2, -2k - 3, k)$

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Q5.

$$\frac{8+4x-x^2-x^3}{(x+4)^2(x^2+4)} = \frac{A}{(x+4)} + \frac{B}{(x+4)^2} + \frac{(Cx+D)}{(x^2+4)}$$

$$\therefore\ A(x+4)(x^2+4) + B(x^2+4) + (Cx+D)(x+4)^2 = 8+4x-x^2-x^3$$

Let $x=-4$:

$$B((-4)^2+4) = 8-16-16-(-64)$$

$$20B = 40$$

$$\Rightarrow\ B=2$$

Let $x=0$:

$$A(4)(4) + B(4) + (D)(4)^2 = 8$$

$$16A + 2\times4 + 16D = 8$$

$$16A = -16D$$

$$A = -D \quad \text{— (1)}$$

Let $x=1$:

$$A(5)(5)+B(5)+(C+D)(5)^2 = 8+4-1-1$$

$$25A + 2\times5 + 25C+25D = 10$$

$$25A + 25C + 25D = 0$$

$$A + C + D = 0$$

(Subst $A=-D$)

$$A + C + (-A) = 0$$

$$\Rightarrow\ C=0$$

Let $x=-1$:

$$A(3)(5) + B(5) + (-C+D)(3)^2 = 8-4-1+1$$

$$15A + 10 - 9C + 9D = 4$$

$$15A + 10 - 0 - (9A) = 4$$

$$6A = -6$$

$$A = -1 \quad \Rightarrow\ D=1$$

Q5.

a) $$\frac{8+4x-x^2-x^3}{(x+4)^2(x^2+4)} = \frac{-1}{(x+4)} + \frac{2}{(x+4)^2} + \frac{1}{x^2+4}$$

4

b) $$\int \frac{8+4x-x^2-x^3}{(x+4)^2(x^2+4)}\,dx = \int_0^2 \left(\frac{2}{(x+4)^2} - \frac{1}{(x+4)} + \frac{1}{x^2+4}\right)dx$$

$$= \int_0^2 \left(2(x+4)^{-2} - \frac{1}{(x+4)} + \frac{1}{(x^2+2^2)}\right)dx$$

$$= \left[\frac{2(x+4)^{-1}}{-1} - \ln|x+4| + \frac{1}{2}\tan^{-1}\left|\frac{x}{2}\right|\right]_0^2$$

$$= \left[\frac{-2}{(x+4)} - \ln|x+4| + \frac{1}{2}\tan^{-1}\left|\frac{x}{2}\right|\right]_0^2$$

$$= \left(\frac{-2}{(2+4)} - \ln|2+4| + \frac{1}{2}\tan^{-1}\left|\frac{2}{2}\right|\right) - \left(\frac{-2}{4} - \ln|4| + \frac{1}{2}\tan^{-1}0\right)$$

$$= \left(\frac{-1}{3} - \ln|6| + \frac{1}{2}\times\left(\frac{\pi}{4}\right)\right) - \left(-\frac{1}{2} - \ln|4| + 0\right)$$

$$= -\frac{1}{3} - \ln|6| + \frac{\pi}{8} + \frac{1}{2} + \ln|4|$$

$$= \frac{1}{2} - \frac{1}{3} + \ln\left|\frac{4}{6}\right| + \frac{\pi}{8}$$

$$= \frac{1}{6} + \ln\left|\frac{2}{3}\right| + \frac{\pi}{8}$$

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Q6. $\boxed{1+\sqrt{3}, \quad 1+\frac{1}{\sqrt{3}}}$

(a) $a = 1+\sqrt{3}$

$$d = \left(1+\frac{1}{\sqrt{3}}\right) - (1+\sqrt{3})$$

$$= 1+\frac{1}{\sqrt{3}} - 1 - \sqrt{3}$$

$$= \frac{1}{\sqrt{3}} - \sqrt{3}\left(\frac{\sqrt{3}}{\sqrt{3}}\right)$$

$$= \frac{1}{\sqrt{3}} - \frac{3}{\sqrt{3}}$$

$$\therefore \quad d = \frac{-2}{\sqrt{3}}$$

$$S_n = \frac{n}{2}\left[2a+(n-1)d\right]$$

$$S_6 = \frac{6}{2}\left[2(1+\sqrt{3}) + 5 \times \frac{-2}{\sqrt{3}}\right]$$

$$= 3\left[2+2\sqrt{3} - \frac{10}{\sqrt{3}}\right]$$

$$= 6+6\sqrt{3} - 10\sqrt{3}$$

$$= 6-4\sqrt{3}$$

$$= 2(3-2\sqrt{3})$$

4

b) $$r = \frac{1+\frac{1}{\sqrt{3}}}{1+\sqrt{3}} = \frac{\sqrt{3}+1}{\sqrt{3}} \div \frac{1+\sqrt{3}}{1} = \frac{\cancel{\sqrt{3}+1}}{\sqrt{3}} \times \frac{1}{\cancel{1+\sqrt{3}}} = \frac{1}{\sqrt{3}}$$

As $-1<r<1$ then S_∞ exists

$$\boxed{S_\infty = \frac{a}{1-r}}$$

Must Rationalise Denominator

$$S_\infty = \frac{(1+\sqrt{3})}{\left(1-\frac{1}{\sqrt{3}}\right)} = \frac{(1+\sqrt{3})}{\left(\frac{\sqrt{3}-1}{\sqrt{3}}\right)} = \frac{\sqrt{3}(1+\sqrt{3})}{(\sqrt{3}-1)} = \frac{\sqrt{3}+3}{(\sqrt{3}-1)} \times \frac{(\sqrt{3}+1)}{(\sqrt{3}+1)}$$

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$$= \frac{3+\sqrt{3}+3\sqrt{3}+3}{(3-1)} = \frac{6+4\sqrt{3}}{2}$$

$$= 3+2\sqrt{3} \text{ As Required.}$$

Q7. <u>Method 1</u>

$z = (-1-3i)$

$z^2 = (-1-3i)(-1-3i) = 1+3i+3i+9i^2 = -8+6i$

$z^3 = (z \times z^2) = (-1-3i)(-8+6i) = 8-6i+24i-18i^2 = 26+18i$

$$\boxed{f(z) = z^3 - z^2 + 4z + k \quad (\text{if factor } R=0)}$$

Then $z^3 - z^2 + 4z + k = 0$

$(26+18i) - (-8+6i) + 4(-1-3i) + k = 0$

$26 + 18i + 8 - 6i - 4 - 12i + k = 0$

$30 + k = 0$

$\therefore \quad k = -30$

<u>Method 2</u>

$$\begin{array}{c|cccc} (-1-3i) & 1 & -1 & 4 & k \\ & & (-1-3i) & (-7+9i) & 30 \\ \hline & 1 & (-2-3i) & (-3+9i) & k+30 \end{array}$$

$(-1-3i)(-2-3i)$
$= 2+9i+9i^2$
$= -7+9i$

$(-1-3i)(-3+9i)$
$= 3-9i+9i-27i^2$
$= 30$

As $R=0 \Rightarrow k+30=0$

$\therefore k=-30$

(OR)

Q7

If $z = (-1-3i)$ then conjugate is $\bar{z} = (-1+3i)$

2 factors are therefore $(z + 1 + 3i)$ & $(z + 1 - 3i)$

Multiplying

$$(z+1+3i)(z+1-3i)$$
$$= z^2 + 2z + 1 - 9i^2$$
$$= z^2 + 2z + 10$$

	z	$+1$	$+3i$
z	z^2	$+z$	$\cancel{+3iz}$
$+1$	$+z$	$+1$	$\cancel{+3i}$
$-3i$	$\cancel{-3iz}$	$\cancel{-3i}$	$-9i^2$

Dividing to find remaining factor gives:

$$\begin{array}{r|l} & z - 3 \\ \hline z^2+2z+10 & z^3 - z^2 + 4z - 30 \\ & z^3 + 2z^2 + 10z \\ \hline & \quad -3z^2 - 6z - 30 \\ & \quad -3z^2 - 6z - 30 \\ \hline & \qquad - \end{array}$$

$\rightarrow$ $z^3 - z^2 + 4z - 30 = 0$

$(z-3)(z+1+3i)(z+1-3i) = 0$

$\downarrow \quad \downarrow \quad \downarrow$

$z = 3$; $z = -1-3i$ & $z = -1+3i$

are 3 roots

$$f(z) = z^3 - z^2 + 4z - 30$$
$$= (z-3)(z^2+2z+10)$$

Where $a = -3$; $b = 2$ & $c = 10$

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Q8. a) $f(x) = x^2 e^{\sin^{-1}(1-x)}$

$u = x^2 \qquad v = e^{\sin^{-1}(1-x)}$

$u' = 2x \qquad v' = -1 \cdot \dfrac{1}{\sqrt{1-(1-x)^2}} \cdot e^{\sin^{-1}(1-x)}$

$= \dfrac{-e^{\sin^{-1}(1-x)}}{\sqrt{2x-x^2}}$

$$\boxed{f'(x) = u'v + uv'}$$

$$\therefore f'(x) = 2xe^{\sin^{-1}(1-x)} - \frac{x^2 e^{\sin^{-1}(1-x)}}{\sqrt{2x-x^2}}$$

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$$= xe^{\sin^{-1}(1-x)}\left(2 - \frac{x}{\sqrt{2x-x^2}}\right)$$

$$= xe^{\sin^{-1}(1-x)}\left(\frac{2\sqrt{2x-x^2} - x}{\sqrt{2x-x^2}}\right)$$

$$= \frac{xe^{\sin^{-1}(1-x)}}{\sqrt{x(2-x)}}\left(2\sqrt{x(2-x)} - x\right)$$

$$= \frac{\left(2\sqrt{x(2-x)} - x\right)xe^{\sin^{-1}(1-x)}}{\sqrt{x}\sqrt{2-x}}$$

$$= \frac{\sqrt{x}\left(2\sqrt{x(2-x)} - x\right)e^{\sin^{-1}(1-x)}}{\sqrt{2-x}}$$

Not nec but can simplify further.

8 b)

$$y = 11^x$$

$$\ln|y| = \ln|11^x|$$

$$\ln|y| = x\ln|11|$$

$$\frac{1}{y}\frac{dy}{dx} = \ln|11|$$

$$\frac{dy}{dx} = \ln|11| \times y$$

$$\therefore \frac{dy}{dx} = \ln|11| \times 11^x$$

3

Q9. $\boxed{\sum_{r=0}^{n} 3^r = \frac{3^{n+1}-1}{2}}$

Let $n=1$: $3^0+3^1 = \frac{3^{1+1}-1}{2}$

$1+3 = \frac{3^2-1}{2}$

$4 = 8/2$ ✓ LHS = RHS $\Rightarrow$ true for $n=1$ ✓

Assume true for $n=k$: $\sum_{r=0}^{n=k} 3^r = \frac{3^{k+1}-1}{2}$

Consider for $n=(k+1)$: $\sum_{r=0}^{n=k+1} 3^r = \frac{3^{k+1}-1}{2} + 3^{(k+1)}$

$= \frac{3^{k+1}-1}{2} + \frac{2.(3^{k+1})}{2}$

$= \frac{3^{k+1}-1+2.3^{k+1}}{2}$

$= \frac{3.3^{k+1}-1}{2}$

$= \frac{3^{k+2}-1}{2}$

$= \frac{3^{(k+1)+1}-1}{2}$

5

Statement:

As true for $n=1$, assumed true for $n=k$ and by Proof of Mathematical Induction also true for $n=k+1$, must be true $\forall n \in \mathbb{N}$.

Q10. $\frac{dy}{dx} = \frac{xy^3}{\sqrt{x^2+1}}$

$$\int \frac{dy}{y^3} = \int \frac{x\,dx}{\sqrt{x^2+1}}$$

Let $u = x^2+1$

$\frac{du}{dx} = 2x$

$\therefore dx = \frac{du}{2x}$

$$\int y^{-3}\,dy = \int \frac{x}{\sqrt{u}} \cdot \frac{du}{2x}$$

$$\int y^{-3}\,dy = \frac{1}{2}\int u^{-1/2}\,du$$

$$\frac{y^{-2}}{-2} = \frac{1}{\cancel{2}}\left(\frac{u^{1/2}}{\cancel{1/2}}\right) + C$$

$$\frac{1}{-2y^2} = \sqrt{u} + C$$

$$\frac{1}{-2(\sqrt{u}+C)} = y^2$$

$$\therefore\ y = \pm\sqrt{\frac{1}{-2(\sqrt{x^2+1}+C)}}$$

If $y=1;\ x=0$:

$$1 = \pm\sqrt{\frac{1}{-2(\sqrt{1}+C)}}$$

(Squaring) $1 = \frac{1}{-2(1+C)}$

$-2(1+C) = 1$

$-2-2C = 1$

$-2C = 3$

$\therefore\ C = -3/2$

General Solution

$$y = \pm\sqrt{\frac{1}{-2(\sqrt{x^2+1}+C)}}$$

Particular Solution ($C = -\frac{3}{2}$)

$$y = \pm\frac{1}{\sqrt{-2\sqrt{(x^2+1)}-2C}}$$

$$y = \pm\frac{1}{\sqrt{-2\sqrt{(x^2+1)}-2\left(-\frac{3}{2}\right)}}$$

$$y = \pm\frac{1}{\sqrt{3-2\sqrt{(x^2+1)}}}$$

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Q11. $y = e^{3x}$ $\quad x = 1$ & $x = 3$

$$\text{Volume} = \int_1^3 \pi y^2\,dx$$

$$= \int_1^3 \pi (e^{3x})^2\,dx$$

$$= \pi \int_1^3 e^{6x}\,dx$$

$$= \pi \left[\frac{e^{6x}}{6}\right]_1^3$$

$$= \frac{\pi}{6}\left[e^{18} - e^{6}\right]$$ ← (fine to stop here)

OR $$= \frac{\pi e^6}{6}(e^{12} - 1)$$ or $\frac{263}{1250}$

4

Q12. $y = x - 3 + \frac{16}{x+3}$

a) Vertical Asymptote : $x = -3$

Non-Vertical Asymptote : $y = x - 3$

2

b) $y = x - 3 + 16(x+3)^{-1}$

$$\frac{dy}{dx} = 1 - 16(x+3)^{-2} = 1 - \frac{16}{(x+3)^2}$$

Stat Pts when $\frac{dy}{dx} = 0$:

$$1 - \frac{16}{(x+3)^2} = 0$$

$$1 = \frac{16}{(x+3)^2}$$

$$(x+3)^2 = 16$$

$$x + 3 = \pm 4$$

$$\therefore x = -3 \pm 4$$

4

So $x = -7$ & $x = 1$

If $x = 1$ $y = 1 - 3 + \frac{16}{4}$

$= -2 + 4$

$\therefore y = 2$

$\Rightarrow (1, 2)$

If $x = -7$ $y = -7 - 3 + \frac{16}{-4}$

$= -10 - 4$

$\therefore y = -14$

$\Rightarrow (-7, -14)$

c)

$(x,y) \to \lvert(x,y)\rvert$	$\to \lvert(x,y)\rvert - 2$
$(1,2) \to (1,2)$	$\to (1,0)$
$(-7,-14) \to (-7,14)$	$\to (-7,12)$

$\therefore (1,0)$ & $(-7,12)$ satisfy $y = \left| x - 3 + \frac{16}{x+3} \right| - 2$

2