Q1.

$$\boxed{(x+y)^n = \sum_{r=0}^{n} \binom{n}{r} (x)^{n-r} (y)^r}$$

$$\left(3u^2 - \frac{5}{v^3}\right)^6 = \sum_{r=0}^{6} \binom{6}{r} (3u^2)^{6-r} \left(\frac{-5}{v^3}\right)^r$$

$$= \underbrace{\sum_{r=0}^{6} \binom{6}{r} (3)^{6-r} . (-5)^r}_{\text{Coefficient}} . \underbrace{(u^2)^{6-r} (v^{-3})^r}_{\text{Term}}$$

Coefficient

$$\binom{6}{r} (3)^{6-r} (-5)^r$$

$r=4$ $\quad C = \binom{6}{4} (3)^{6-4} (-5)^4$

$$= \frac{6!}{4!\,2!} \times 3^2 \times (-5)^4$$

$$= \frac{6 \times 5 \times \cancel{4!}}{\cancel{4!}\,2!} \times 9 \times 625$$

$$= 15 \times 9 \times 625$$

$$= \underline{\underline{84375}}$$

Term

$$(u^2)^{6-r} (v^{-3})^r = u^4 v^{-12}$$

$$u^{12-2r} . v^{-3r} = u^4 . v^{-12}$$

So either $u^{12-2r} = u^4$

$$12 - 2r = 4$$

$$-2r = -8$$

$$\underline{\underline{r = 4}}$$

OR

$$v^{-3r} = v^{-12}$$

$$-3r = -12$$

$$\underline{\underline{r = 4}}$$

5

Q2.

$x = t^5 - 9t^3$ $\qquad$ $y = 3t^2$

$\frac{dx}{dt} = 5t^4 - 27t^2$ $\qquad$ $\frac{dy}{dt} = 6t$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{6t}{5t^4 - 27t^2} = \frac{6}{5t^3 - 27t} \text{ or } \frac{6}{t(5t^2 - 27)}$$

(a) At $(0, 27) \Rightarrow x = 0$ & $y = 27$

$x = t^5 - 9t^3$

$\therefore t^5 - 9t^3 = 0$

$t^3(t^2 - 9) = 0$

$t^3 = 0 \qquad t^2 - 9 = 0$

$t = 0 \qquad t^2 = 9$

$t = \pm 3$

$y = 3t^2$

$\therefore 3t^2 = 27$

$t^2 = 9$

$t = \pm 3$

Must satisfy both x & y $\therefore t = \pm 3$

So can now find gradient $m = dy/dx$

$$\boxed{m = \frac{dy}{dx} = \frac{6}{5t^3 - 27t}}$$

At $t = 3$ $\quad m = \frac{6}{5(3)^3 - 27(3)} = \frac{6}{135 - 81} = \frac{6}{54}$

$\therefore m = 1/9$

& At $t = -3$ $\quad m = \frac{6}{5(-3)^3 - 27(-3)} = \frac{6}{-135 + 81} = \frac{6}{-54} = -1/9$

Q2a) Continued

At $t = 3$: $m = \frac{1}{9}$ & $(0, 27)$

$$(y-27) = \frac{1}{9}(x-0)$$

$$y - 27 = \frac{1}{9}x$$

$$9y - 243 = x$$

$$\therefore \quad x - 9y + 243 = 0$$

$$\left(\text{or } y = \tfrac{1}{9}x + 27\right)$$

At $t = -3$: $m = -\frac{1}{9}$ & $(0, 27)$

$$(y-27) = -\frac{1}{9}(x-0)$$

$$9y - 243 = -x$$

$$\therefore \quad x + 9y - 243 = 0$$

6

Q2(b)

$$\frac{dy}{dx} = \frac{6}{5t^3 - 27t} = 6(5t^3-27t)^{-1}$$

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}} = \frac{-6(5t^3-27t)^{-2}.(15t^2-27)}{5t^4 - 27t^2}$$

$$= \frac{\frac{-6}{(5t^3-27t)^2} . [3(5t^2-9)]}{t^2(5t^2-27)}$$

$$= \frac{-18(5t^2-9)}{t^2(5t^2-27) \times (5t^3-27t)^2}$$

$$= \frac{-18(5t^2-9)}{t^2(5t^2-27)(t(5t^2-27))^2}$$

$$= \frac{-18(5t^2-9)}{t^2(5t^2-27)(t^2(5t^2-27)^2)}$$

$\rightarrow$ Need to express as requested

Q2b) Continued:

$$\frac{d^2y}{dx^2} = \frac{-18(5t^2-9)}{t^2(5t^2-27)(t^2(5t^2-27)^2)}$$

$$= \frac{-90t^2+162}{t^4(5t^2-27)^3}$$

$$= \frac{162-90t^2}{t(t(5t^2-27))^3}$$

$$= \frac{162-90t^2}{t(5t^3-27t)^3}$$

Although nicer format, must give in the layout specified. So need to realise $t^4 = t \times t^3$ and manipulate brackets to achieve format required

3

$\boxed{\text{Where, } a = 162;\ b = -90;\ c = 5\ \&\ d = -27}$

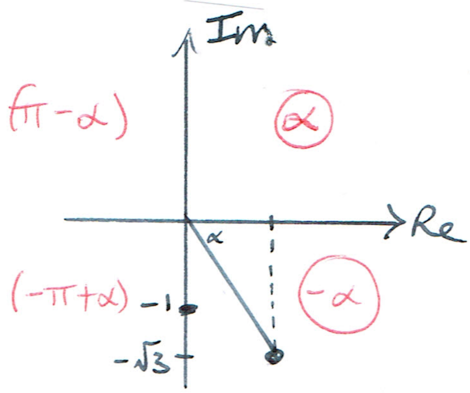
Q3 a) $1-\sqrt{3}i$

$|r| = \sqrt{(1)^2+(-\sqrt{3})^2}$

$= \sqrt{1+3}$

$= \sqrt{4}$

$\therefore |r| = 2$



1st Quad:

$\arg(z) = \tan^{-1}|b/a|$

$= \tan^{-1}\left|\frac{-\sqrt{3}}{1}\right|$

$= \tan^{-1}\left(\frac{\sqrt{3}}{1}\right)$

$= \frac{\pi}{3}$

As in 4th Quad $(-\alpha) \Rightarrow \theta = -\frac{\pi}{3}$

$\boxed{\text{De Moivre's } z = r(\cos\theta + i\sin\theta)}$

2

$\therefore \quad z = 1-\sqrt{3}i = 2\left(\cos\left(-\frac{\pi}{3}\right) + i\sin\left(-\frac{\pi}{3}\right)\right)$

(b)

$z^n = (1-\sqrt{3}i)^n = \left[2\left(\cos\left(-\frac{\pi}{3}\right)+i\sin\left(-\frac{\pi}{3}\right)\right)\right]^n$

$= 2^n\left(\cos\left(-\frac{n\pi}{3}\right)+i\sin\left(-\frac{n\pi}{3}\right)\right)$ Mark Here

$\therefore \quad (1-\sqrt{3}i)^n = 2^n\left(\cos\left(\frac{n\pi}{3}\right) - i\sin\left(\frac{n\pi}{3}\right)\right)$ 1

Recall Odd/Even Signs

As Cos is even function & Sine is Odd
So $\cos\left(-\frac{\pi}{3}\right) = \cos\left(\frac{\pi}{3}\right)$ & $\sin\left(\frac{-\pi}{3}\right) = -\sin\left(\frac{\pi}{3}\right)$

Q3(c)

$z = 1 - \sqrt{3}i \Rightarrow$ A Conjugate Pair exists.

$\therefore \ \bar{z} = 1 + \sqrt{3}i$

So factors are $(z - 1 + \sqrt{3}i)$ & $(z - 1 - \sqrt{3}i)$

Multiplying gives: $(z - 1 + \sqrt{3}i)(z - 1 - \sqrt{3}i)$

	z	-1	$+\sqrt{3}i$
z	z^2	$-z$	$\cancel{+\sqrt{3}iz}$
-1	$-z$	$+1$	$\cancel{-\sqrt{3}i}$
$-\sqrt{3}i$	$\cancel{-\sqrt{3}iz}$	$\cancel{+\sqrt{3}i}$	$-3i^2$

$= z^2 - z - z + 1 - 3i^2$

$= z^2 - 2z + 1 + 3$

$= z^2 - 2z + 4$

Need to use long division to find remaining roots

$z^6 + 4(1-i)z^3 + p + qi = 0$

$$\begin{array}{r l}
& z^4 + 2z^3 + (-4-4i)z + (-8-8i) \\
z^2 - 2z + 4 \,\big) & z^6 + 0z^5 + 0z^4 + (4-4i)z^3 + 0z^2 + 0z + (p+qi) \\
& z^6 - 2z^5 + 4z^4 \\
\hline
& 2z^5 - 4z^4 + (4-4i)z^3 + 0z^2 + 0z + (p+qi) \\
& 2z^5 - 4z^4 + 8z^3 \\
\hline
& (-4-4i)z^3 + 0z^2 + 0z + (p+qi) \\
& (-4-4i)z^3 + (8+8i)z^2 + (-16-16i)z + 0 \\
\hline
& (-8-8i)z^2 + (16+16i)z + (p+qi) \\
& (-8-8i)z^2 + (16+16i)z + (-32-32i) \\
\hline
& (p+32) + (q+32)i
\end{array}$$

If $(p+32) = 0$ & $(q+32) = 0$

$\therefore p = -32$ $\quad\therefore q = -32$

5

(As if solution $= 0 \Rightarrow$ No Remainder exists.)

Q3(c) Alternative Method 2 (As z^6 is unusual)

If $z = 1-\sqrt{3}i$

then, $z^2 = (1-\sqrt{3}i)(1-\sqrt{3}i) = 1 - \sqrt{3}i - \sqrt{3}i + 3i^2$

$= 1 - 2\sqrt{3}i - 3$

$= \underline{-2 - 2\sqrt{3}i}$

So, $z^3 = z \times z^2$

$= (1-\sqrt{3}i)(-2-2\sqrt{3}i)$

$= -2 - \cancel{2\sqrt{3}i} + \cancel{2\sqrt{3}i} + 6i^2$

$= \underline{-8}$

and, $z^6 = (z^3)^2 = (-8)^2 = \underline{64}$

So $z^6 + 4(1-i)z^3 + p + qi = 0$

$(64) + (4-4i)(-8) + p + qi = 0$

5

$64 - 32 + 32i + p + qi = 0$

$(32+p) + (32+q)i = 0$

As perfect solution No REMAINDER EXISTS so Set Real = 0 & Imaginary = 0 & Solve

$(32+p) = 0$ & $(32+q) = 0$

$\therefore \underline{\underline{p = -32}}$ $\therefore \underline{\underline{q = -32}}$

Q3 (c) If use DeMoivres in Polar form Method 3

As $\boxed{z^n = 2^n\left(\cos\left(\frac{n\pi}{3}\right) - i\sin\left(\frac{n\pi}{3}\right)\right) \text{ from (b)}}$

$$z^6 + 4(1-i)z^3 + p + qi = 0$$

$$\left(2^6\left(\cos\left(\frac{6\pi}{3}\right) - i\sin\left(\frac{6\pi}{3}\right)\right)\right) + 4(1-i)\left(2^3\left(\cos\left(\frac{3\pi}{3}\right) - i\sin\left(\frac{3\pi}{3}\right)\right)\right) + p + qi = 0$$

$$64\left(\cos 2\pi - i\sin 2\pi\right) + (4-4i)\left(8\left(\cos\pi - \sin\pi\right)\right) + p + qi = 0$$

$$64\left(1 - i(0)\right) + (4-4i)\left(8(-1-0i)\right) + p + qi = 0$$

$$64 + (4-4i)(-8) + p + qi = 0$$

$$64 - 32 + 32i + p + qi = 0$$

$$32 + 32i + p + qi = 0$$

$\therefore$ $p + 32 = 0$ & $qi + 32i = 0$

$\underline{\underline{p = -32}}$

$qi = -32i$

$\therefore \underline{\underline{q = -32}}$

5

Remember
Set Real = Real
& Im = Im

Q4.

$$\begin{aligned}2x - y + \alpha z &= 1\\ x - y + 2z &= -3\\ -x + 2y - 3z &= 2\end{aligned} \longrightarrow \begin{array}{c}\left(\begin{array}{ccc|c} 2 & -1 & \alpha & 1\\ 1 & -1 & 2 & -3\\ -1 & 2 & -3 & 2\end{array}\right)\end{array}\begin{array}{c} r_1\\ r_2\\ r_3\end{array}$$

$$\begin{array}{c} r_1\\ 2r_2\\ 2r_3\end{array}\left(\begin{array}{ccc|c} 2 & -1 & \alpha & 1\\ 2 & -2 & 4 & -6\\ -2 & 4 & -6 & 4\end{array}\right)\begin{array}{c} r_1\\ r_2\\ r_3\end{array} \rightarrow \begin{array}{c} r_1\\ r_2-r_1\\ r_3+r_1\end{array}\left(\begin{array}{ccc|c} 2 & -1 & \alpha & 1\\ 0 & -1 & 4-\alpha & -7\\ 0 & 3 & \alpha-6 & 5\end{array}\right)\begin{array}{c} r_1\\ r_2\\ r_3\end{array}$$

4

$$\begin{array}{c} r_1\\ 3r_2\\ r_3\end{array}\left(\begin{array}{ccc|c} 2 & -1 & \alpha & 1\\ 0 & -3 & 12-3\alpha & -21\\ 0 & 3 & \alpha-6 & 5\end{array}\right)\begin{array}{c} r_1\\ r_2\\ r_3\end{array} \rightarrow \begin{array}{c} r_1\\ r_2\\ r_2+r_3\end{array}\left(\begin{array}{ccc|c} 2 & -1 & \alpha & 1\\ 0 & -3 & 12-3\alpha & -21\\ 0 & 0 & 6-2\alpha & -16\end{array}\right)$$

If $\alpha = 3$: $(6-2\alpha)z = -16$

$\therefore 0z = -16$

$0 \neq -16$!

As $0x + 0y + 0z = -16$

Impossible! $\Rightarrow$ INCONSISTENT

So NO SOLUTION EXISTS

1

If $\alpha = -13$

$(6-2\alpha)z = -16$

$(6+26)z = -16$

$32z = -16$

$\therefore z = -\frac{1}{2}$

$-3y + (12-3\alpha)z = -21$

$-3y + (12+39)\left(-\frac{1}{2}\right) = -21$

$-3y + 51 \times \left(-\frac{1}{2}\right) = -21$

$-3y = -21 + \frac{51}{2}$

$-3y = \frac{9}{2}$

$\therefore y = -3/2$

$2x - y + \alpha z = 1$

$2x + \frac{3}{2} + \left(-13 \times -\frac{1}{2}\right) = 1$

$2x + \frac{3}{2} + \frac{13}{2} = 1$

$2x + 8 = 1$

$2x = -7$

$\therefore x = \frac{-7}{2}$

1

$\therefore$ Unique Solution at $\left(\frac{-7}{2}, -\frac{3}{2}, \frac{-1}{2}\right)$

Q5 a) $2 + x - 2x^2 - x^3$

$= -x^3 - 2x^2 + x + 2$

$$\begin{array}{c|cccc} 1 & -1 & -2 & 1 & 2 \\ & & -1 & -3 & -2 \\ \hline & -1 & -3 & -2 & \underline{\underline{0}} \end{array} \checkmark$$

As $R = 0 \Rightarrow x = 1$ is a solution & $(x-1)$ is a factor.

$\therefore \; 2 + x - 2x^2 - x^3 = (x-1)(-x^2 - 3x - 2)$

$= -(x-1)(x^2 + 3x + 2)$

$= -(x-1)(x+2)(x+1)$

1

b)

$$\begin{array}{r|l} & \quad -1 \\ \hline -x^3 - 2x^2 + x + 2 & x^3 - 3x^2 - 2x + 10 \\ & x^3 + 2x^2 - x - 2 \\ \hline & \quad -5x^2 - x + 12 \end{array}$$

Thus $\displaystyle\int_2^3 \frac{x^3 - 3x^2 - 2x + 10}{2 + x - 2x^2 - x^3}\,dx = \int_2^3 \left(-1 + \frac{(-5x^2 - x + 12)}{-(x-1)(x+1)(x+2)}\right) dx$

$= \displaystyle\int_2^3 \left(-1 + \frac{5x^2 + x - 12}{(x-1)(x+1)(x+2)}\right) dx$

$\downarrow$

Need to split using Partial Fractions

Q5(b)

$$\frac{5x^2+x-12}{(x-1)(x+1)(x+2)} = \frac{A}{(x-1)} + \frac{B}{(x+1)} + \frac{C}{(x+2)}$$

$$\therefore \quad 5x^2+x-12 = A(x+1)(x+2) + B(x-1)(x+2) + C(x-1)(x+1)$$

Let $x=1$: $5+1-12 = A(2)(3)$

$$-6 = 6A$$

$$\therefore \quad \underline{\underline{A=-1}}$$

Let $x=-1$: $5-1-12 = B(-2)(1)$

$$-8 = -2B$$

$$\therefore \quad \underline{\underline{B=4}}$$

Let $x=-2$: $20-2-12 = C(-3)(-1)$

$$6 = 3C$$

$$\therefore \quad \underline{\underline{C=2}}$$

$$\therefore \quad \frac{5x^2+x-12}{(x-1)(x+1)(x+2)} = \frac{-1}{(x-1)} + \frac{4}{(x+1)} + \frac{2}{(x+2)}$$

$$\therefore \int_2^3 \left(-1 + \frac{(5x^2+x-12)}{(x-1)(x+1)(x+2)}\right) dx = \int_2^3 \left(-1 - \frac{1}{(x-1)} + \frac{4}{(x+1)} + \frac{2}{(x+2)}\right) dx$$

$$= \left[-x - \ln|x-1| + 4\ln|x+1| + 2\ln|x+2|\right]_2^3$$

$$= \left[-x + \ln|(x+1)^4| + \ln|(x+2)^2| - \ln|x-1|\right]_2^3$$

$$= \left[-x + \ln\left|\frac{(x+1)^4(x+2)^2}{(x-1)}\right|\right]_2^3$$

8

$$= \left(-3 + \ln\left|\frac{(4)^4(5)^2}{(2)}\right|\right) - \left(-2 + \ln\left|\frac{(3)^4(4)^2}{(1)}\right|\right)$$

$$= -3 + \ln|3200| + 2 - \ln|1296| = \ln\left|\frac{200}{81}\right| - 1$$

06. AH 2011/12 Units 1 & 2 Prelim.

$a+10,\ a+5,\ a+2,\ \ldots$ Geometric Sequence

$$r = \frac{a+5}{a+10} = \frac{a+2}{a+5} \Rightarrow (a+5)(a+5) = (a+2)(a+10)$$

$$\cancel{a^2} + 10a + 25 = \cancel{a^2} + 12a + 20$$

$$10a + 25 = 12a + 20$$

$$-2a = -5$$

$$a = \frac{5}{2}$$

[care not u_1] So first term, $u_1 = a+10 = \frac{5}{2} + 10 = \frac{25}{2}$

If $r = \dfrac{a+5}{a+10} = \dfrac{\left(\frac{5}{2}\right)+5}{\left(\frac{5}{2}\right)+10} = \dfrac{\frac{15}{2}}{\frac{25}{2}} = \dfrac{15}{25} = \dfrac{3}{5}$ 4

b) $\boxed{u_n = ar^{n-1}}$ 2

$u_6 = ar^5$

But

$\boxed{\text{a represents 1st term so confusing as not 'a' in question}}$

$$= \left(\frac{25}{2}\right)\cdot\left(\frac{3}{5}\right)^5$$

$$= \frac{25}{2} \times \left(\frac{243}{3125}\right)$$

$$= \frac{6075}{6250}$$

$$= \frac{243}{250}$$

(c) $S_\infty = \dfrac{a}{1-r} = \dfrac{\left(25/2\right)}{1-3/5}$ 3

$$= \frac{25}{2} \div \frac{2}{5}$$

$$= \frac{25}{2} \times \frac{5}{2}$$

$$= \frac{125}{4} \text{ or } 31\tfrac{1}{4} \ (\text{or } 31 \cdot 25)$$

Q7. $\ln\left|\frac{y^2}{x^3}\right| = y\sqrt{x} - e^{1-x-y}$

$\ln|y^2| - \ln|x^3| = y x^{1/2} - \left(e^{(1-x)} \cdot e^{-y}\right)$

$\underbrace{2\ln|y| - 3\ln|x|}_{\bullet} = \underbrace{(y \cdot x^{1/2})}_{*} - \underbrace{(e^{1-x} \cdot e^{-y})}_{*}$

$\boxed{y x^{1/2}}$

$u = y \qquad v = x^{1/2}$

$u' = dy/dx \qquad v' = \frac{1}{2}x^{-1/2}$

$\boxed{e^{1-x} \cdot e^{-y}}$

$u = e^{1-x} \qquad v = e^{-y}$

$u' = -e^{1-x} \qquad v' = -e^{-y}\frac{dy}{dx}$

$\frac{2}{y} \cdot \frac{dy}{dx} - \frac{3}{x} = \left(x^{1/2} \cdot \frac{dy}{dx} + \frac{1}{2}x^{-1/2} \cdot y\right) - \left[\left(-e^{1-x} \cdot e^{-y}\right) + \left(-e^{-y} \cdot \frac{dy}{dx} \cdot e^{1-x}\right)\right]$

$\frac{2}{y} \cdot \frac{dy}{dx} - \frac{3}{x} = \sqrt{x}\frac{dy}{dx} + \frac{y}{2\sqrt{x}} + e^{1-x} \cdot e^{-y} + e^{1-x} \cdot e^{-y} \cdot \frac{dy}{dx}$

$\left(\frac{2}{y} - \sqrt{x} - e^{1-x-y}\right)\frac{dy}{dx} = \left(\frac{3}{x} + \frac{y}{2\sqrt{x}} + e^{1-x-y}\right)$

4

$\therefore \quad \frac{dy}{dx} = \dfrac{\left(\frac{3}{x} + \frac{y}{2\sqrt{x}} + e^{1-x-y}\right)}{\left(\frac{2}{y} - \sqrt{x} - e^{1-x-y}\right)}$

Accepted in Marking Scheme as this

OR $\quad \frac{dy}{dx} = \dfrac{\left(\frac{6 + \sqrt{x}y + 2xe^{1-x-y}}{2x}\right)}{\left(\frac{2 - \sqrt{x}y - ye^{1-x-y}}{y}\right)} = \dfrac{y\left(6 + \sqrt{x}y + 2xe^{1-x-y}\right)}{2x\left(2 - \sqrt{x}y - ye^{1-x-y}\right)}$

Reduce & Simplify to

Q7 (Continued)

$$\frac{dy}{dx} = \frac{y\left(6+\sqrt{x}\,y + 2x\,e^{1-x-y}\right)}{2x\left(2-\sqrt{x}\,y - y\,e^{1-x-y}\right)}$$

$\boxed{\begin{matrix}\text{At } (4,-3) \\ (x, y)\end{matrix}}$

$$\frac{dy}{dx} = \frac{(-3)\left(6+\sqrt{4}(-3)+2(4)e^{1-4-(-3)}\right)}{2(4)\left(2-\sqrt{4}(-3) - (-3)e^{1-(4)-(-3)}\right)}$$

$$\therefore \frac{dy}{dx} = \frac{-3\times\left(6+2\times-3+2\times4\times e^{1-4+3}\right)}{(2\times4)\left(2-2\times-3+3e^{1-4+3}\right)}$$

$$= \frac{-3\left(6-6+8e^{0}\right)}{8\left(2+6+3e^{0}\right)}$$

$$= \frac{-3\,(\cancel{8})}{\cancel{8}\,(8+3)}$$

$$\therefore m = \frac{-3}{11}$$

2

Q8 a)

$$\sec^2\theta - 1 = \frac{1}{\cos^2\theta} - 1 = \frac{1}{\cos^2\theta} - \frac{\cos^2\theta}{\cos^2\theta} = \frac{1-\cos^2\theta}{\cos^2\theta} = \frac{\sin^2\theta}{\cos^2\theta} = \tan^2\theta$$ • 1

Q8 b)

$$x = \frac{3}{4}\sec\theta$$

Need to address $\int_{\sqrt{3}/2}^{3/2}$ and change to terms of θ rather than x

$$x = \sqrt{3}/2 \Rightarrow \frac{\sqrt{3}}{2} = \frac{3}{4}\sec\theta$$

$$\frac{4\times\sqrt{3}}{2\times 3} = \sec\theta$$

$$\frac{2}{\sqrt{3}} = \frac{1}{\cos\theta}$$

When $x = \frac{\sqrt{3}}{2}$ $\quad \cos\theta = \frac{\sqrt{3}}{2}$

$\therefore\ \theta = \pi/3$ *

$$x = 3/2 \Rightarrow \frac{3}{2} = \frac{3}{4}\sec\theta$$

$$\frac{4}{2} = \sec\theta$$

$$\frac{2}{1} = \frac{1}{\cos\theta}$$

When $x = \frac{3}{2}$ $\quad \cos\theta = 1/2$

$\therefore\ \theta = \pi/6$ *

Need to change dx to $d\theta$

If $x = \frac{3}{4}\sec\theta = \frac{3}{4\cos\theta} = \frac{3}{4}(\cos\theta)^{-1}$

then $\frac{dx}{d\theta} = -\frac{3}{4}(\cos\theta)^{-2}\cdot -\sin\theta$

$$dx = \frac{3\sin\theta}{4\cos^2\theta}\,d\theta = \frac{3}{4}\left(\frac{\sin\theta}{\cos\theta}\right)\left(\frac{1}{\cos\theta}\right)d\theta = \frac{3}{4}\tan\theta\sec\theta\,d\theta$$ *

AH 2011/12 Units 1 & 2 Prelim

Q8 b)

$$\int_{\frac{\sqrt{3}}{2}}^{\frac{3}{2}} \frac{\sqrt{(16x^2-9)}}{x}\,dx = \int_{\pi/6}^{\pi/3} \frac{\sqrt{16\left(\frac{3}{4}\sec\theta\right)^2-9}}{\left(\cancel{\frac{3}{4}\sec\theta}\right)} \cdot \cancel{\frac{3}{4}\sec\theta}\tan\theta\,d\theta$$

$\int_{\frac{\sqrt{3}}{2}}^{\frac{3}{2}} \rightarrow \int_{\pi/6}^{\pi/3}$

& $dx = \frac{3}{4}\sec\theta\tan\theta\,d\theta$

from previous page

$$= \int_{\pi/6}^{\pi/3} \sqrt{\cancel{16}\left(\frac{9}{\cancel{16}}\sec^2\theta\right)-9}\;.\tan\theta\,d\theta$$

$$= \int_{\pi/6}^{\pi/3} \sqrt{9\sec^2\theta-9}\;.\tan\theta\,d\theta$$

$$= 3\int_{\pi/6}^{\pi/3} \sqrt{\sec^2\theta-1}\;.\tan\theta\,d\theta$$

$$= 3\int_{\pi/6}^{\pi/3} \tan^2\theta\,d\theta$$

we know the derivative of $\tan x$ is $\sec^2 x$

so $\int \sec^2 x\,dx = \tan x + C$

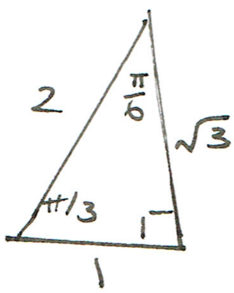
$\therefore$ Change $\tan^2\theta$ to $(\sec^2\theta - 1)$ to $\int$ Integ.

$$= 3\int_{\pi/6}^{\pi/3} (\sec^2\theta - 1)\,d\theta$$

$$= 3\Big[\tan\theta - \theta\Big]_{\pi/6}^{\pi/3}$$

6

$$= 3\left(\left(\tan\frac{\pi}{3} - \frac{\pi}{3}\right) - \left(\tan\frac{\pi}{6} - \frac{\pi}{6}\right)\right)$$



$$= 3\left(\frac{\sqrt{3}}{1} - \frac{\pi}{3} - \frac{1}{\sqrt{3}} + \frac{\pi}{6}\right)$$

$$= (3\sqrt{3} - \sqrt{3}) - \pi + \frac{\pi}{2}$$

$$= 2\sqrt{3} - \frac{\pi}{2} = \frac{1}{2}(4\sqrt{3} - \pi)$$

Q9. $\boxed{8^n - 2^n \text{ is divisible by } 6}$

Let $n=1$: $8^1 - 2^1 = 8 - 2 = 6 = 6 \times 1$ ✓ true for $n=1$

Assume true for $n=k$: $8^k - 2^k = 6m$ (where m is a positive integer $m \in \mathbb{Z}^+$ & $8^k - 2^k$ is a divisible by 6)

Consider $n=k+1$

$$8^{k+1} - 2^{k+1} = 8.8^k - 2.2^k$$
$$= 8.8^k - 2.2^k + (8.2^k - 8.2^k)$$
$$= (8.8^k - 8.2^k) - 2.2^k + 8.2^k$$
$$= 8(8^k - 2^k) + 8.2^k - 2.2^k$$
$$= 8(8^k - 2^k) + 2^k(8-2)$$
$$= 8(6m) + 2^k(6)$$
$$= 6\underline{(8m + 2^k)}$$

Statement:

As true for $n=1$, and as assumed true for $n=k$ by Proof of Mathematical Induction true for $n=k+1$.

Thus true $\forall n \in \mathbb{N}$

5

Q10)
(a) $f(x) = \dfrac{x-3}{x+2} = \underbrace{1 - \dfrac{5}{x+2}}$

$$\begin{array}{r|l} & 1 \\ \hline x+2 & x-3 \\ & x+2 \\ \hline & -5 \end{array}$$

$\therefore$ Vertical Asymptote $x = -2$

Horizontal Asymptote $y = 1$

3

(b) $f(x) = 1 - 5(x+2)^{-1}$

$\therefore f'(x) = 5(x+2)^{-2}$

$= \dfrac{5}{(x+2)^2}$

As Stat Pt when $f'(x) = 0$

$\dfrac{5}{(x+2)^2} = 0$

$5 \neq 0$

Impossible! $\Rightarrow$ No Stat Pts Exist

2

(c) $f''(x) = -10(x+2)^{-3}$

$= \dfrac{-10}{(x+2)^3}$

As POI when $f''(x) = 0$

$\dfrac{-10}{(x+2)^3} = 0$

$-10 \neq 0$ Impossible!

$\therefore$ No Points of Inflexion

2

3

(d) Asymptotes of Inverse

Inverse $\Rightarrow$ $x = 1$ & $y = -2$

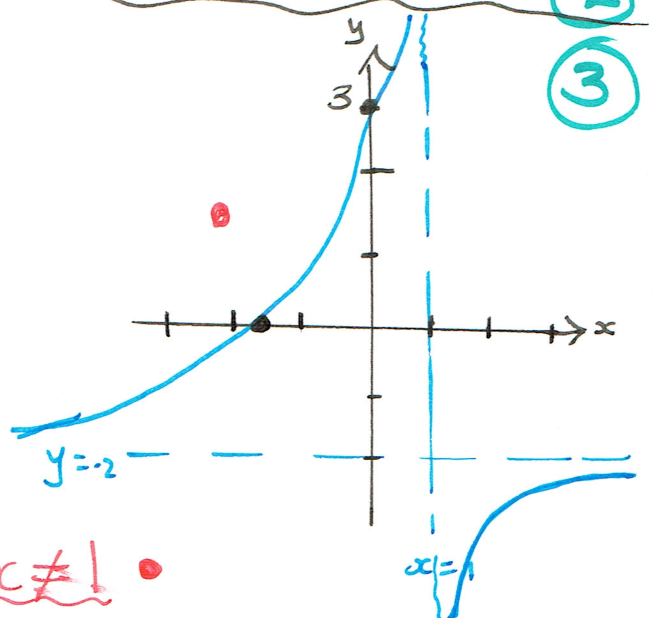
Original $f(x)$ Cuts at

$x = 0$ $y = \frac{-3}{2}$ $(0, -3/2) \Rightarrow$ Inverse $(-3/2, 0)$

&

$y = 0$ $x - 3 = 0$ $(3, 0) \Rightarrow (0, 3)$

$x = 3$



For $f^{-1}(x)$, Domain is $x \neq 1$