AH 2004

Q1.

$f(x) = \cos^2 x \, e^{\tan x}$

$\boxed{f'(x) = u'v + uv'}$

$u = (\cos x)^2 \qquad v = e^{\tan x}$

$u' = 2(\cos x)^1 \cdot -\sin x \qquad v' = \sec^2 x \, e^{\tan x}$

$= -2\sin x \cos x$

$$f'(x) = (-2\sin x \cos x) \cdot e^{\tan x} + \cos^2 x \times \sec^2 x \, e^{\tan x}$$

$$= e^{\tan x}\left[-2\sin x \cos x + \cos^2 x \times \left(\frac{1}{\cos^2 x}\right)\right]$$

$$= e^{\tan x}\left[1 - 2\sin x \cos x\right]$$

3

$\boxed{\text{or } \; e^{\tan x}(1 - \sin 2x)}$

$$f'(x) = e^{\tan x}(1 - 2\sin x \cos x)$$

$$f'\left(\tfrac{\pi}{4}\right) = e^{\tan(\frac{\pi}{4})}\left(1 - 2\sin\left(\tfrac{\pi}{4}\right)\cos\left(\tfrac{\pi}{4}\right)\right)$$

$$= e^1\left(1 - 2 \times \frac{1}{\sqrt{2}} \times \frac{1}{\sqrt{2}}\right)$$

$$= e(1 - 1)$$

1

$$= \underline{\underline{0}}$$

Q1 b) $g(x) = \dfrac{\tan^{-1}(2x)}{1+4x^2}$

$u = \tan^{-1}(2x) \qquad v = 1+4x^2$

$u' = \dfrac{1}{1+(2x)^2} \cdot 2 \qquad v' = 8x$

$= \dfrac{2}{1+4x^2}$

$$\boxed{g'(x) = \frac{u'v - uv'}{v^2}}$$

$$g'(x) = \frac{\frac{2}{\cancel{(1+4x^2)}} \cdot \cancel{(1+4x^2)} - 8x\tan^{-1}(2x)}{(1+4x^2)^2}$$

$$= \frac{2 - 8x\tan^{-1}(2x)}{(1+4x^2)^2}$$

$$= \frac{2(1-4x\tan^{-1}(2x))}{(1+4x^2)^2}$$

3

Q2. $(a^2-3)^4 = \binom{4}{0}(a^2)^4(-3)^0 + \binom{4}{1}(a^2)^3(-3)^1 + \binom{4}{2}(a^2)^2(-3)^2 + \binom{4}{3}(a^2)^1(-3)^3 + \binom{4}{4}(a^2)^0(-3)^4$

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1

$= (1 \times a^8 \times 1) + (4 \times a^6 \times -3) + (6 \times a^4 \times 9) + (4 \times a^2 \times -27) + (1 \times 1 \times 81)$

Bin coeffs

Powers

$\therefore (a^2-3)^4 = a^8 - 12a^6 + 54a^4 - 108a^2 + 81$

3

Q3. $x = 5\cos\theta$ & $y = 5\sin\theta$

$$\frac{dx}{d\theta} = -5\sin\theta \qquad \frac{dy}{d\theta} = 5\cos\theta$$

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{5\cos\theta}{-5\sin\theta} = -\frac{\cos\theta}{\sin\theta} \text{ or } -\cot\theta$$

2

When $\theta = \frac{\pi}{4}$ $\quad m = \frac{dy}{dx} = \frac{-\cos(\pi/4)}{\sin(\pi/4)} = \frac{-1/\sqrt{2}}{1/\sqrt{2}} = -1$

$$x = 5\cos\theta = 5\times\cos(\pi/4) = 5/\sqrt{2} \text{ or } \frac{5\sqrt{2}}{2}$$

$$y = 5\sin\theta = 5\times\sin(\pi/4) = 5/\sqrt{2} \text{ or } \frac{5\sqrt{2}}{2}$$

$M = -1$ & $(5/\sqrt{2}, 5/\sqrt{2})$

$$(y-b) = m(x-a)$$

$$\left(y - \frac{5}{\sqrt{2}}\right) = -1\left(x - \frac{5}{\sqrt{2}}\right)$$

3

$$y - \frac{5}{\sqrt{2}} = -x + \frac{5}{\sqrt{2}}$$

$$x + y = \frac{5}{\sqrt{2}} + \frac{5}{\sqrt{2}}$$

$$x + y = \frac{10}{\sqrt{2}} \times \left(\frac{\sqrt{2}}{\sqrt{2}}\right) = \frac{10\sqrt{2}}{2}$$

$$\therefore\ x + y = 5\sqrt{2}$$

Q4. $z = \underset{\sim\sim\sim}{1+2i}$

$z^2 = (1+2i)^2 = 1 + 4i + 4i^2 = \underset{\sim\sim\sim}{-3+4i}$

$z^3 = z \times z^2 = (1+2i)(-3+4i)$

$= -3 + 4i - 6i + 8i^2$

$= \underset{\sim\sim\sim}{-11 - 2i}$

$z^2(z+3) = z^3 + 3z^2$

$= (-11-2i) + 3(-3+4i)$

$= -11 - 2i - 9 + 12i$ 2

$= \underset{\sim\sim\sim}{-20+10i}$

$z^3 + 3z^2 - 5z + 25 = (-20+10i) - 5(1+2i) + 25$

$= -20 + 10i - 5 - 10i + 25$

$= \underline{\underline{0}}$ 2

As $z = 1+2i$ has $R=0 \Rightarrow$ is a solution of the equation.

If $z = 1+2i$ is a root $\Rightarrow \underline{\bar{z} = 1-2i}$ is also a root (conjugate pair).

$(z-1-2i)(z-1+2i) = z^2 - z - z + 1 - 4i^2$

$= \underset{\sim\sim\sim}{z^2 - 2z + 5}$ 2

	z	-1	$-2i$
z	z^2	$-z$	$\cancel{-2iz}$
-1	$-z$	$+1$	$\cancel{+2i}$
$+2i$	$\cancel{+2iz}$	$\cancel{-2i}$	$-4i^2$

$$
\begin{array}{r|l}
 & z + 5 \\
\hline
z^2 - 2z + 5 & z^3 + 3z^2 - 5z + 25 \\
 & \underline{z^3 - 2z^2 + 5z} \\
 & \quad 5z^2 - 10z + 25 \\
 & \quad \underline{5z^2 - 10z + 25}
\end{array}
$$

$\therefore$ $\boxed{\text{Roots are } z = 1 \pm 2i;\ z = -5}$

Q5. $$\frac{1}{x^2-x-6} = \frac{1}{(x-3)(x+2)} = \frac{A}{(x-3)} + \frac{B}{(x+2)}$$

$$A(x+2) + B(x-3) = 1$$

Let $x = 3$: $\quad 5A = 1 \Rightarrow A = 1/5$

Let $x = -2$: $\quad -5B = 1 \Rightarrow B = -1/5$

2

$$\therefore \frac{1}{x^2-x-6} = \frac{1/5}{(x-3)} + \frac{-1/5}{(x+2)} = \frac{1}{5(x-3)} - \frac{1}{5(x+2)}$$

$$\int_0^1 \frac{1}{x^2-x-6}\,dx = \int_0^1 \left(\frac{1}{5(x-3)} - \frac{1}{5(x+2)}\right) dx$$

$$= \frac{1}{5}\int_0^1 \left(\frac{1}{(x-3)} - \frac{1}{(x+2)}\right) dx$$

$$= \frac{1}{5}\Big[\ln|x-3| - \ln|x+2|\Big]_0^1$$

$$= \frac{1}{5}\left[\ln\left|\frac{x-3}{x+2}\right|\right]_0^1$$

4

$$= 1/5\left(\ln\left|\frac{1-3}{1+2}\right| - \ln\left|\frac{-3}{2}\right|\right)$$

$$= 1/5\left(\ln\left|\frac{-2}{3}\right| - \ln\left|\frac{-3}{2}\right|\right)$$

If $|-2| = 2$

then $\ln|-2/3| = \ln|2/3|$

$$= \frac{1}{5}\left(\ln\left(\frac{2/3}{3/2}\right)\right)$$

$$= 1/5 \ln|4/9|$$

(≈ -0.162)

Q6

A-C. Rotation of $\pi/2$

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} = \begin{pmatrix} \cos\frac{\pi}{2} & -\sin\frac{\pi}{2} \\ \sin\frac{\pi}{2} & \cos\frac{\pi}{2} \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$ 2

Reflection of θ ie. $(x, y) \to (x, -y) \Rightarrow \theta = 180°$

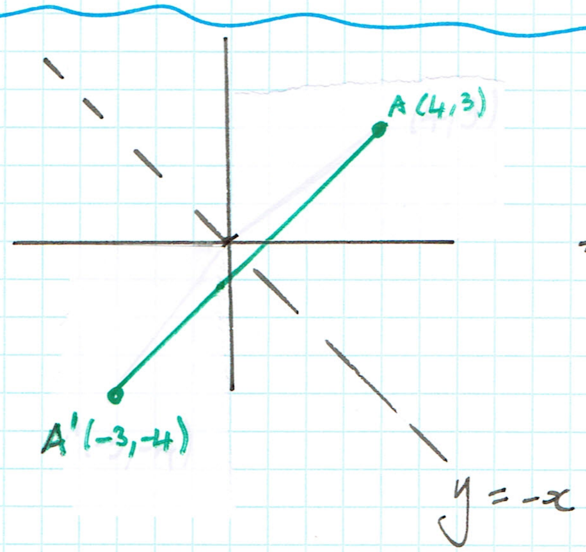
$$\begin{pmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{pmatrix} = \begin{pmatrix} \cos 360 & \sin 360 \\ \sin 360 & -\cos 360 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$ 1

$$\boxed{M_2 M_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}}$$ 1

Given a point $A(x, y)$ with transformation $M_2 M_1$

$$\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -y \\ -x \end{pmatrix}$$

ie. $A(4, 3) \to A'(-3, -4)$



$\Rightarrow$ Geometrically this reflects in line $y = -x$ 1

Q7.

$f(x) = e^x \sin x$

Let $g(x) = e^x$	$g(0) = e^0 = 1$	Let $h(x) = \sin x$	$h(0) = 0$
$g'(x) = e^x$	$g'(0) = 1$	$h'(x) = \cos x$	$h'(0) = 1$
$g''(x) = e^x$	$g''(0) = 1$	$h''(x) = -\sin x$	$h''(0) = 0$
$g'''(x) = e^x$	$g'''(0) = 1$	$h'''(x) = -\cos x$	$h'''(0) = -1$
$g^{iv}(x) = e^x$	$g^{iv}(0) = 1$	$h^{iv}(x) = \sin x$	$h^{iv}(0) = 0$
		$h^{v}(x) = \cos x$	$h^{v}(0) = 1$

$f(x) = g(x)h(x)$ Not nec to call $g(x)$ & $h(x)$ but helps distinguish between them

$$\boxed{f(x) \simeq f(0) + f'(0)\frac{x}{1!} + f''(0)\frac{x^2}{2!} + f'''(0)\frac{x^3}{3!} + \dots + f^n(0)\frac{x^n}{n!}}$$

$$g(x) = e^x \simeq 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

$$h(x) = \sin x \simeq 0 + \frac{x}{1!} + 0\frac{x^2}{2!} - \frac{x^3}{3!} + 0\frac{x^4}{4!} + \frac{x^5}{5!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} \dots$$

$\therefore f(x) = g(x)h(x)$

$f(x) = e^x \sin x$

	1	$+\,x$	$+\,\frac{x^2}{2}$	$+\,\frac{x^3}{6}$
x	x	$+\,x^2$	$+\,x^3/2$	—
$-\frac{x^3}{6}$	$-\frac{x^3}{6}$	—	—	—
$\frac{x^5}{120}$	—	—	—	—

If asked for 1st 3 Non-Zero terms, no need to find every term

$\therefore$ First 3 terms of $f(x) = e^x \sin x = x + x^2 + \left(\frac{x^3}{2} - \frac{x^3}{6}\right) = x + x^2 + \frac{x^3}{3} + \dots$

5

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(9)

Q9.

$x = (u-1)^2$

$\frac{dx}{du} = 2(u-1)$

$dx = 2(u-1).du$

& If $x = (u-1)^2$

$\sqrt{x} = (u-1)$

$\Rightarrow u = (1+\sqrt{x})$

$$\int \frac{1}{(1+\sqrt{x})^3}dx = \int \frac{1}{\left(1+\sqrt{(u-1)^2}\right)^3} \cdot 2(u-1)du$$

$$= \int \frac{2(u-1).du}{(1+u-1)^3}$$

$$= 2\int \frac{u-1}{u^3}du$$

(All in terms of u)

$$= 2\int (u^{-2} - u^{-3})du$$

$$= 2\left[\frac{u^{-1}}{-1} - \frac{u^{-2}}{-2}\right] + c$$

$$= -\frac{2}{u} + \frac{2}{2u^2} + c$$

$$= \frac{1}{u^2} - \frac{2}{u} + c$$

$$= \frac{1-2u}{u^2} + c$$

5

$$= \frac{1-2(\sqrt{x}+1)}{(1+\sqrt{x})^2} + c$$

$$= \frac{-2\sqrt{x}-1}{(1+\sqrt{x})^2} + c$$

Not nec. to have as 1 fraction for final mark → $\left(\frac{1}{(1+\sqrt{x})^2} - \frac{2}{(1+\sqrt{x})}\right) + c$

Q10. $f(x) = x^4 \sin 2x$

$$f(-x) = (-x)^4 \cdot \sin 2(-x)$$
$$= x^4 \times -\sin 2x$$
$$= -x^4 \sin 2x$$
$$= -f(x)$$

As $f(-x) = -f(x)$
$\Rightarrow$ function is ODD

3

Q11. $\boxed{V = \int_a^b \pi y^2\, dx}$

Between $x = 0$ & $x = 1$
when $y = e^{-2x}$

$$V = \int_0^1 \pi (e^{-2x})^2\, dx$$
$$= \pi \int_0^1 (e^{-2x} \times e^{-2x})\, dx$$
$$= \pi \int_0^1 e^{-4x}\, dx$$
$$= \pi \left[\frac{e^{-4x}}{-4} \right]_0^1$$
$$= \pi \left[\frac{e^{-4}}{-4} - \frac{e^0}{-4} \right)$$
$$= \pi \left(\frac{-e^{-4}}{4} + \frac{1}{4} \right)$$
$$= \frac{\pi}{4}\left(1 - e^{-4}\right) \text{ units}^3$$

5

or $\frac{\pi}{4}\left(1 - \frac{1}{e^4}\right)$

or 0.771

Q12. $$\boxed{\frac{d^n}{dx^n}(xe^x) = (x+n)e^x}\ ,\ \forall n \geq 1$$

$$\boxed{\begin{array}{ll} u = x & v = e^x \\ u' = 1 & v' = e^x \end{array}}$$

Let $n=1$

$$\frac{d}{dx}(xe^x) = e^x + xe^x$$
$$= e^x.(1+x)$$
$$= (1+x).e^x$$

✓ true for $n=1$

Assume true for $n=k$

$$\frac{d^k}{dx^k}(xe^x) = (x+k)e^x$$

where k is a positive integer.

Consider $n=k+1$

$$\frac{d^{k+1}}{dx^{k+1}}(xe^x) = \frac{d}{dx}\left(\frac{d^k}{dx^k}(xe^x)\right)$$

$$\boxed{\begin{array}{ll} u = x+k & v = e^x \\ u' = 1 & v' = e^x \end{array}}$$

$$= \frac{d}{dx}((x+k)e^x)$$
$$= 1.e^x + (x+k)e^x$$
$$= e^x(1+(x+k))$$
$$= (x+(k+1))e^x$$

As required.

As true for $n=1$, assumed true for $n=k$ and by proof of mathematical induction also true for $n=k+1$. Thus

$$\frac{d^n}{dx^n}(xe^x) = (x+n)e^x,\ \forall n \geq 1 \text{ is true } \forall n \in \mathbb{N}.$$

(5)

Q13. $f(x) = \dfrac{x-3}{x+2}, \; x \neq -2.$

$$\begin{array}{r|l} & 1 \\ \hline x+2 & x-3 \\ & x+2 \\ \hline & -5 \end{array}$$

$\therefore \; f(x) = 1 - \dfrac{5}{(x+2)}$

a) Vertical Asymptote : $x = -2$ & Non-Vertical : $y = 1$ (3)

b) $f(x) = 1 - 5(x+2)^{-1}$

$f'(x) = 5(x+2)^{-2} = \dfrac{5}{(x+2)^2}$

Stat Pts when $f'(x) = 0$: $\dfrac{5}{(x+2)^2} = 0$

$\Rightarrow \; 5 \neq 0$! $\boxed{\text{Impossible!}}$

As $5 \neq 0 \Rightarrow$ No Stationary Points Exist. (2)

c) $f''(x) = -10(x+2)^{-3} = \dfrac{-10}{(x+2)^3}$

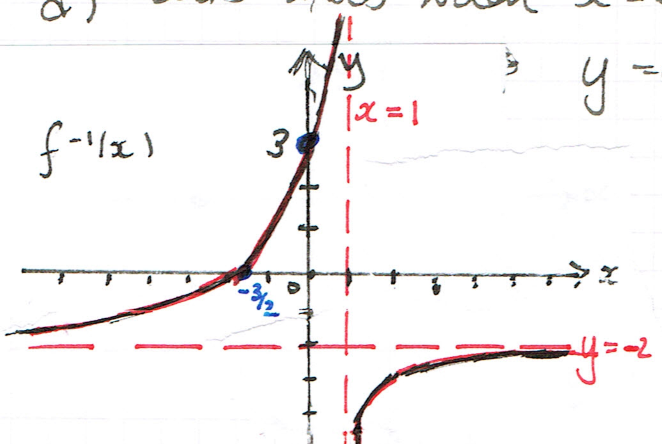
Points of Inflexion when $f''(x) = 0 \Rightarrow \dfrac{-10}{(x+2)^3} = 0$ (2)

As impossible $\Rightarrow$ No Points of Inflexion exist $\quad -10 \neq 0$!

d) Cuts axes when $x = 0$: $y = \dfrac{-3}{2} \Rightarrow (0, -3/2)$

$y = 0$: $0 = x - 3 \Rightarrow (3, 0)$

$x = 3$ (3)



$f(x)$	$\rightarrow$	$f^{-1}(x)$
$y=1$	$\rightarrow$	$x=1$
$x=-2$	$\rightarrow$	$y=-2$
$(0, -3/2)$	$\rightarrow$	$(-3/2, 0)$
$(3, 0)$	$\rightarrow$	$(0, 3)$

• Asymptotes
• Sketch
• $x \neq 1$
Domain

AH 2004

Q16. $8 + 11 + 14 + \dots + 56 \rightarrow$ $\boxed{\text{So can find } n^{th} \text{ value.}}$

$a = 8$ & $d = 3$

$$\boxed{U_n = a + (n-1)d}$$

$$56 = 8 + (n-1) \times 3$$

$$56 = 8 + 3n - 3$$

$$\therefore \quad 51 = 3n$$

$$\Rightarrow \quad n = 17$$

$$\boxed{S_n = \frac{n}{2}\left[2a + (n-1)d\right]}$$

(2)

$$\left.\begin{matrix} a = 8 \\ d = 3 \\ n = 17 \end{matrix}\right\} \quad S_{17} = \frac{17}{2}\left[2 \times 8 + (17-1) \times 3\right]$$

$$= \frac{17}{2}\left[16 + 16 \times 3\right]$$

$$= \frac{17 \times 64}{2}$$

$$\therefore S_{17} = 544$$

---

b) $\boxed{\begin{matrix} S_3 = 266 \\ a = 2 \end{matrix}}$ $\quad \boxed{a + ar + ar^2 = 266}$

$$2 + 2r + 2r^2 = 266$$

$$r^2 + r + 1 = 133$$

$$r^2 + r - 132 = 0$$

$$(r - 11)(r + 12) = 0$$

(3)

$$r = 11 \quad \& \quad r = -12$$

As positive terms, $r > 0 \Rightarrow r = 11$

Q16 c) AH 2004

Seq Ⓐ $u_1 = a$; $d = 2$

$$S_4 = a + (a+d) + (a+2d) + (a+3d)$$
$$= 4a + 6d$$
$$= 4a + 12 \quad \text{—} ①$$

Seq Ⓑ $u_1 = a$; $r = 2$

$$S_4 = a + ar + ar^2 + ar^3$$
$$= a + 2a + 4a + 8a$$
$$= 15a \quad \text{—} ②$$

$$S_4 = S_4 \Rightarrow 15a = 4a + 12$$
$$11a = 12$$
$$\therefore a = \frac{12}{11}$$

3

Ⓐ $S_n = \frac{n}{2}[2a + (n-1)d]$

$$S_n = \frac{n}{2}\left[2\left(\frac{12}{11}\right) + (n-1)\times 2\right]$$
$$= \frac{12}{11}n + n^2 - n$$
$$= n^2 + \frac{n}{11}$$
$$\therefore S_A = \frac{11n^2 + n}{11}$$

Ⓑ $S_n = \frac{a(1-r^n)}{1-r}$

$$S_n = \frac{\left(\frac{12}{11}\right)(1-2^n)}{(1-2)}$$
$$= -\frac{12}{11}(1-2^n)$$
$$\therefore S_B = \frac{12}{11}(2^n - 1)$$

$\boxed{B > 2A}$

$$\frac{12}{\cancel{11}}(2^n - 1) > 2\left(\frac{11n^2+n}{\cancel{11}}\right) \quad \times 11$$
$$12(2^n - 1) > 2(11n^2 + n) \quad \div 2$$
$$6(2^n - 1) > 11n^2 + n \quad \div 6$$
$$2^n - 1 > \frac{11n^2 + n}{6}$$
$$2^n > \frac{11n^2 + n + 6}{6}$$

Valid Strategy (Not nec. to rearrange formula)

2

Try $n = 6$: $2^6 = 64 < 136$

$n = 7$: $2^7 = 128 > 92$ ✓

$\therefore n = 7$ is first time $B > 2A$.