AH 2003

(1)

A1

$$f(x) = x(1+x)^{10}$$

$$\boxed{f'(x) = u'v + uv'}$$

$$f'(x) = (1+x)^{10} + x(10(1+x)^9)$$

$$= (1+x)^{10} + 10x(1+x)^9$$

$$= [(1+x) + 10x](1+x)^9$$

$$= (1+11x)(1+x)^9$$

(3)

b)

$$y = 3^x$$

$$\ln|y| = \ln|3^x|$$

$$\ln|y| = x\ln|3|$$

(must bring x to front for mark)

$$\frac{1}{y}\cdot\frac{dy}{dx} = \ln|3|$$

$$\frac{dy}{dx} = \ln|3| \times y$$

(3)

$$\therefore \frac{dy}{dx} = \ln|3| \times 3^x$$

A2. $u_k = 11 - 2k \quad (k \geq 1)$

$$\left.\begin{aligned} u_1 &= 11 - 2 = 9 \\ u_2 &= 11 - 4 = 7 \\ u_3 &= 11 - 6 = 5 \end{aligned}\right\} \quad 9, 7, 5 \Rightarrow a = 9 \ \& \ d = -2$$

$$\begin{aligned} S_n &= \frac{n}{2}\left[2a + (n-1)d\right] \\ &= \frac{n}{2}\left[2(9) + (n-1) \times -2\right] \\ &= \frac{n}{2}\left[18 - 2n + 2\right] \\ &= \frac{n}{2}\left[20 - 2n\right] \\ &= n(10 - n) \\ &= 10n - n^2 \end{aligned}$$

3

$$\boxed{S_n = 21 \ \& \ S_n = 10n - n^2}$$

$$21 = 10n - n^2$$

$$n^2 - 10n + 21 = 0$$

$$(n-7)(n-3) = 0$$

$$n = 7 \ ; \ n = 3$$

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3

A3. $y^3 + 3xy = 3x^2 - 5$

$$3y^2\frac{dy}{dx} + \left(3y + 3x\frac{dy}{dx}\right) = 6x$$

3xy her

$u = 3x \quad v = y$

$u' = 3 \quad v' = \frac{dy}{dx}$

$u'v + uv'$

$\left(3y + 3x\frac{dy}{dx}\right)$

$$(3y^2 + 3x)\frac{dy}{dx} = 6x - 3y$$

$$\frac{dy}{dx} = \frac{6x - 3y}{3y^2 + 3x} = \frac{2x - y}{y^2 + x}$$

At A(2,1) $\quad m = \frac{2(2) - (1)}{(1)^2 + (2)} = \frac{4-1}{1+2} = \frac{3}{3} = 1$

$$(y - 1) = 1(x - 2)$$

$$y - 1 = x - 2$$

$$y = x - 1$$

4

A4 $\quad |z + i| = 2$

$$|(x + iy) + i| = 2$$

$$|(x) + i(y + 1)| = 2$$

$$\sqrt{x^2 + (y+1)^2} = 2$$

$$x^2 + (y+1)^2 = 4$$

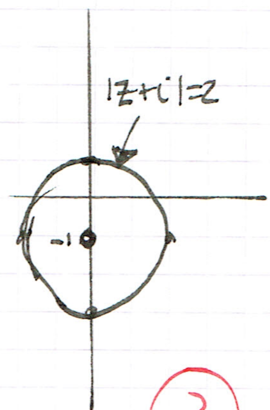
$\Rightarrow (0, -1), \ r = 2$

OR $\quad |z + i| = 2$

$\downarrow$

Centre $(0, -1)$

$r = 2$



- Circle
- Centre + radius correct
- Circumference highlighted

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④

A5.

$$\int_0^{\pi/2} \frac{\cos\theta}{(1+\sin\theta)^3} \cdot d\theta$$

$$x = 1+\sin\theta$$

$$\frac{dx}{d\theta} = \cos\theta$$

$$dx = \cos\theta \, d\theta$$

$$\therefore d\theta = \frac{dx}{\cos\theta}$$

$\theta = 0$:

$$x = 1+\sin 0$$

$$x = 1$$

$\theta = \frac{\pi}{2}$:

$$x = 1+\sin\frac{\pi}{2}$$

$$x = 1+1$$

$$x = 2$$

$$= \int_1^2 \frac{\cos\theta}{(x)^3} \cdot \frac{dx}{\cos\theta}$$

$$= \int_1^2 \frac{dx}{x^3}$$

$$= \int_1^2 x^{-3}\, dx$$

$$= \left[\frac{x^{-2}}{-2}\right]_1^2$$

$$= \left[\frac{1}{-2(x)^2}\right]_1^2$$

$$= \left(\frac{1}{-2(2)^2}\right) - \left(\frac{1}{-2(1)^2}\right)$$

$$= \frac{1}{-8} + \frac{1}{2}$$

$$= \frac{4}{8} - \frac{1}{8}$$

$$= \frac{3}{8}$$

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A6.
$$\begin{aligned} x + y + 3z &= 1 \\ 3x + ay + z &= 1 \\ x + y + z &= -1 \end{aligned} \rightarrow \begin{array}{c} \left(\begin{array}{ccc|c} 1 & 1 & 3 & 1 \\ 3 & a & 1 & 1 \\ 1 & 1 & 1 & -1 \end{array}\right) \end{array} \begin{array}{c} r_1 \\ r_2 \\ r_3 \end{array}$$

$$\begin{array}{c} 3r_1 \\ r_2 \\ 3r_3 \end{array} \left(\begin{array}{ccc|c} 3 & 3 & 9 & 3 \\ 3 & a & 1 & 1 \\ 3 & 3 & 3 & -3 \end{array}\right) \begin{array}{c} r_1 \\ r_2 \\ r_3 \end{array} \rightarrow \begin{array}{c} r_1 \\ r_2 - r_1 \\ r_3 - r_1 \end{array} \left(\begin{array}{ccc|c} 3 & 3 & 9 & 3 \\ 0 & (a-3) & -8 & -2 \\ 0 & 0 & -6 & -6 \end{array}\right) \begin{array}{c} r_1 \\ r_2 \\ r_3 \end{array}$$

* 2

$$\begin{array}{c} \frac{1}{3}r_1 \\ r_2 \\ -\frac{1}{6}r_3 \end{array} \left(\begin{array}{ccc|c} 1 & 1 & 3 & 1 \\ 0 & (a-3) & -8 & -2 \\ 0 & 0 & 1 & 1 \end{array}\right) \begin{array}{c} r_1 \\ r_2 \\ r_3 \end{array}$$

If $a \neq 3$ $\quad z = 1$

$$\begin{aligned} (a-3)y - 8z &= -2 \\ (a-3)y - 8 &= -2 \\ (a-3)y &= 6 \\ y &= \frac{6}{(a-3)} \end{aligned}$$

$$\begin{aligned} x + y + 3z &= 1 \\ x + \frac{6}{(a-3)} + 3 &= 1 \\ x &= -2 - \frac{6}{(a-3)} \end{aligned}$$

2

If $a = 3$

$$\left(\begin{array}{ccc|c} 1 & 1 & 3 & 1 \\ 0 & 0 & -8 & -2 \\ 0 & 0 & 1 & 1 \end{array}\right)$$

In r_3 $\quad z = 1$

But in r_2 $\quad -8z = -2$

$z = 1/4$ 2

$z \neq 1$ & $1/4$ $\Rightarrow$ Inconsistent results

$\Rightarrow$ No solution exists

* Solutions (-1) for using upper triangular, but would argue with this as most sensible method

A7. $y = \dfrac{x}{1+x^2}$

$$\boxed{\begin{array}{ll} u = x & v = 1+x^2 \\ u' = 1 & v' = 2x \end{array}}$$

$$\boxed{\frac{dy}{dx} = \frac{u'v - uv'}{v^2}}$$

$$\therefore \frac{dy}{dx} = \frac{(1+x^2) - x(2x)}{(1+x^2)^2}$$

$$= \frac{1+x^2-2x^2}{(1+x^2)^2}$$

$$= \frac{1-x^2}{(1+x^2)^2}$$

Stat Pts when $dy/dx = 0 \Rightarrow \dfrac{1-x^2}{(1+x^2)^2} = 0$

4

$$1 - x^2 = 0$$

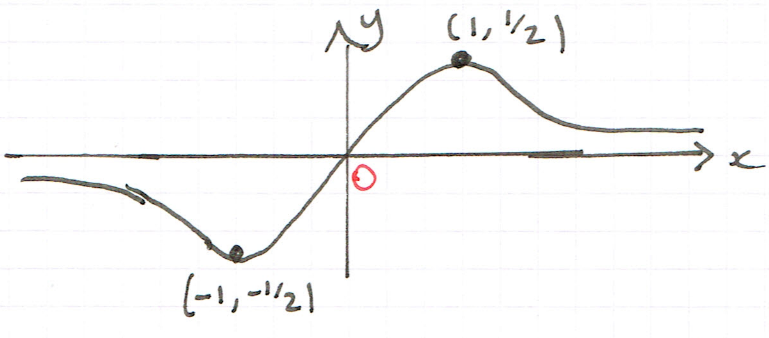
$$1 = x^2$$

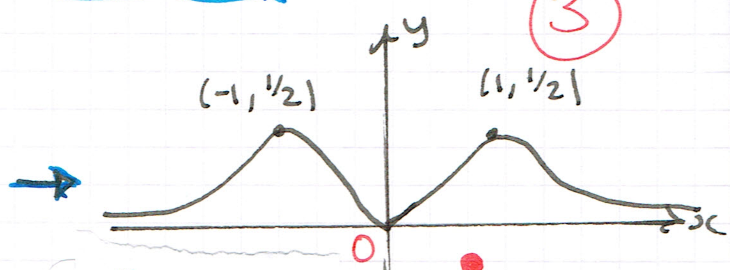
$$\therefore \underline{\underline{x = \pm 1}}$$

If $x = 1$: $y = \dfrac{1}{1+1} = \underline{\underline{1/2}}$ $(1, 1/2)$

If $x = -1$: $y = \dfrac{-1}{1+1} = \underline{\underline{-1/2}}$ $(-1, -1/2)$

3





- Sketch
- $(1, \frac{1}{2}), (-1, \frac{1}{2})$ TPts
- Critical Pt $(0, 0)$ (Discontinuous 'Sharp' Point)

A8.

$p(n) = n^2+n$, where n is a positive integer

(A) Is $p(n)$ always even?

Assume n is odd, so let $n = (2k+1)$ when $p(n)$ is even where k is a positive integer.

$$\begin{aligned}
p(n) &= n^2+n \\
&= (2k+1)^2 + (2k+1) \\
&= (4k^2+4k+1) + (2k+1) \\
&= (4k^2+6k+2) \\
&= 2(2k^2+3k+1) \\
&= 2(2k+1)(k+1)
\end{aligned}$$

4

$\Rightarrow$ Divisible by 2

$\Rightarrow$ $p(n)$ is always even & A is TRUE

(B) Is $p(n)$ always a multiple of 3?

Let $n=1$ $p(1) = 1^2+1 = 2 \neq 3m$

2 is not divisible by 3, so (B) is FALSE

and Conjecture is disproved.

A9. $w = \cos\theta + i\sin\theta$

Then $\dfrac{1}{w} = \dfrac{1}{(\cos\theta + i\sin\theta)} \times \dfrac{(\cos\theta - i\sin\theta)}{(\cos\theta - i\sin\theta)} = \dfrac{\cos\theta - i\sin\theta}{\cos^2\theta - i^2\sin^2\theta}$

$= \dfrac{\cos\theta - i\sin\theta}{(\cos^2\theta + \sin^2\theta)}$

$\dfrac{1}{w} = \underline{\cos\theta - i\sin\theta}$

Alternatively $w = \cos\theta + i\sin\theta$

Then $w^{-1} = (\cos\theta + i\sin\theta)^{-1}$

$= (\cos(-\theta) + i\sin(-\theta))$

$\therefore w^{-1} = \cos\theta - i\sin\theta$

Using De Moivres' & Odd/Even function properties.

$w^k + w^{-k} = (\cos\theta + i\sin\theta)^k + (\cos\theta + i\sin\theta)^{-k}$

$= (\cos(k\theta) + i\sin(k\theta)) + (\cos(-k\theta) + i\sin(-k\theta))$

$= \cos(k\theta) + \cancel{i\sin(k\theta)} + \cos(k\theta) - \cancel{i\sin}(k\theta)$

4

$\therefore \underline{w^k + w^{-k} = 2\cos(k\theta)}$

A9

$$w = \cos\theta + i\sin\theta$$
$$w^{-1} = \cos\theta - i\sin\theta$$
$$w^{k} + w^{-k} = 2\cos k\theta$$

From previous page we found these properties.

$$(w^1 + w^{-1})^4 = (1)(w)^4(w^{-1})^0 + (4)(w)^3(w^{-1})^1 + (6)(w)^2(w^{-1})^2 + (4)(w)^1(w^{-1})^3 + (1)(w)^0(w^{-1})^4$$

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1

$$= (1 \times w^4 \times 1) + (4 \times w^3 \times w^{-1}) + (6 \times w^2 \times w^{-2}) + (4 \times w \times w^{-3}) + (1 \times 1 \times w^{-4})$$

$$= w^4 + 4w^2 + 6 + 4w^{-2} + w^{-4}$$

$$= (w^4 + w^{-4}) + (4w^2 + 4w^{-2}) + 6$$

$$\therefore (w^1 + w^{-1})^4 = (w^4 + w^{-4}) + 4(w^2 + w^{-2}) + 6$$

$$= (2\cos 4\theta) + 4(2\cos 2\theta) + 6$$

$$= 2\cos 4\theta + 8\cos 2\theta + 6$$

5

Also $(w^1 + w^{-1})^4 = (2\cos\theta)^4 = 16\cos^4\theta$

$$\therefore \quad 16\cos^4\theta = 2\cos 4\theta + 8\cos 2\theta + 6$$

$$\cos^4\theta = \frac{2}{16}\cos 4\theta + \frac{8}{16}\cos 2\theta + \frac{6}{16}$$

$$\therefore \quad \cos^4\theta = \frac{1}{8}\cos 4\theta + \frac{1}{2}\cos 2\theta + \frac{3}{8}$$

As Required.

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A10. $\boxed{I_n = \int_0^1 x^n e^{-x}\,dx \quad , n \geq 1}$

a) $I_1 = \int_0^1 x e^{-x}\,dx$

$\boxed{\begin{array}{ll} u = x & v = -e^{-x} \\ u' = 1 & v' = e^{-x} \end{array}}$

$= \left[-xe^{-x}\right]_0^1 - \int_0^1 -e^{-x}\,dx$ $\quad \boxed{\int uv' = uv - \int u'v}$

$= \left[-xe^{-x}\right]_0^1 + \int_0^1 e^{-x}\,dx$

$= \left[-xe^{-x} - e^{-x}\right]_0^1$

$= (-1e^{-1} - e^{-1}) - (0 - e^0)$

3

$\therefore \; I_1 = 1 - 2e^{-1}$

b) $I_n = \int_0^1 x^n e^{-x}\,dx$

$\boxed{\begin{array}{ll} u = x^n & v = -e^{-x} \\ u' = nx^{n-1} & v' = e^{-x} \end{array}}$

$\therefore \; I_n = \left[-x^n e^{-x}\right]_0^1 - \int_0^1 -nx^{n-1}e^{-x}\,dx$ $\quad \boxed{\int uv' = uv - \int u'v}$

$I_n = \left[-x^n e^{-x}\right]_0^1 + n\int_0^1 x^{n-1}e^{-x}\,dx$

$I_n = (-1e^{-1} - 0) + n\int_0^1 x^{n-1}e^{-x}\,dx$

4

$I_n = -e^{-1} + n\,I_{n-1}$

$\therefore \; I_n = nI_{n-1} - e^{-1}$

3

$I_3 = 3I_2 - e^{-1}$

$= 3(2I_1 - e^{-1}) - e^{-1} = 6I_1 - 4e^{-1}$

$= 6(1 - 2e^{-1}) - 4e^{-1}$

$= 6 - 16e^{-1}$

AH 2003 Part A

A11.

$$\frac{dV}{dt} = V(10-V)$$

$$\int \frac{dV}{V(10-V)} = \int dt$$

$$\frac{1}{V(10-V)} = \frac{A}{V} + \frac{B}{(10-V)}$$

$$\therefore\ 1 = A(10-V) + BV$$

Let $V=0$

$$1 = 10A$$

$$\therefore\ A = 1/10$$

Let $V=10$

$$1 = 10B$$

$$\therefore\ B = 1/10$$

$$\int \left(\frac{A}{V} + \frac{B}{(10-V)}\right) dV = \int dt$$

$$\int \left(\frac{1/10}{V} + \frac{1/10}{(10-V)}\right) dV = \int dt$$

$$\frac{1}{10}\int \left(\frac{1}{V} + \frac{1}{(10-V)}\right) dV = \int dt$$

$$\frac{1}{10}\left[\ln|V| + \frac{\ln|10-V|}{-1}\right] = t + c$$

$$\frac{1}{10}\ln|V| - \frac{1}{10}\ln|10-V| = t + c$$

4

As Required.

$$\frac{1}{10}\left(\ln|V| - \ln|10-V|\right) = t + c$$

$$\frac{1}{10}\left(\ln\left|\frac{V}{10-V}\right|\right) = t + c$$

$$\boxed{V(0) = 5 \Rightarrow t = 0,\ V = 5}$$

1

$$\frac{1}{10}\left(\ln\left|\frac{5}{10-5}\right|\right) = 0 + c$$

$$\frac{1}{10}\left(\ln|1|\right) = c \Rightarrow c = 0$$

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A11. $$\frac{1}{10}\left(\ln\left|\frac{v}{10-v}\right|\right) = t + c$$

If $c=0$: $$\frac{1}{10}\left(\ln\left|\frac{v}{10-v}\right|\right) = t$$

$$\therefore \quad \ln\left|\frac{v}{10-v}\right| = 10t$$

$$\frac{v}{(10-v)} = e^{10t}$$

$$v = (10-v)e^{10t}$$

$$v = 10e^{10t} - ve^{10t}$$

2

$$v + ve^{10t} = 10e^{10t}$$

$$(1+e^{10t})v = 10e^{10t}$$

$$\therefore \quad v = \frac{10e^{10t}}{1+e^{10t}}$$

÷ by e^{10t} to simplify to find limit

$$v = \frac{10}{1/e^{10t}+1}$$

2

As $t \to \infty$ $$v \to \frac{10}{\frac{1}{e^{\infty}}+1} = \frac{10}{0+1} = 10$$

$\therefore$ Thus limit tends to 10 as $t \to \infty$