Both Solutions to U3 Prac

Outcome 1

Q1. $\underline{a} = \underline{i} - 2\underline{j} + \underline{k}$

$\underline{b} = 2\underline{i} + \underline{j}$

$$\underline{a} \times \underline{b} = \begin{pmatrix} \underline{i} & \underline{j} & \underline{k} \\ 1 & -2 & 1 \\ 2 & 1 & 0 \end{pmatrix}$$

$$= \begin{vmatrix} -2 & 1 \\ 1 & 0 \end{vmatrix}\underline{i} - \begin{vmatrix} 1 & 1 \\ 2 & 0 \end{vmatrix}\underline{j} + \begin{vmatrix} 1 & -2 \\ 2 & 1 \end{vmatrix}$$

$$= (0-1)\underline{i} - (0-2)\underline{j} + (1-(-4))\underline{k}$$

$$= -\underline{i} + 2\underline{j} + 5\underline{k}$$

3

Q2. $\underline{d} = \vec{AB} = \underline{b} - \underline{a} = \begin{pmatrix} -1 \\ 4 \\ 2 \end{pmatrix} - \begin{pmatrix} 2 \\ -3 \\ 5 \end{pmatrix} = \begin{pmatrix} -3 \\ 7 \\ -3 \end{pmatrix}$

$\boxed{r = a + \lambda d} \Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ -3 \\ 5 \end{pmatrix} + \lambda \begin{pmatrix} -3 \\ 7 \\ -3 \end{pmatrix}$

2

$\therefore \quad x = 2 - 3\lambda \; ; \; y = -3 + 7\lambda \; \& \; z = 5 - 3\lambda$

Q3. $\boxed{r.n = a.n}$ $\quad \begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ 5 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix}$

$$3x + 2y - z = 3 - 4 - 5$$

$$\therefore \quad 3x + 2y - z = -6$$

2

$\boxed{\text{Threshold} = \frac{5}{7}}$

Outcome 2

Q4(a)

$$3A - B + 2C = 3\begin{pmatrix}-4 & 1\\ 2 & 5\end{pmatrix} - \begin{pmatrix}6 & -2\\ -1 & 4\end{pmatrix} + 2\begin{pmatrix}-2 & 4\\ 1 & 8\end{pmatrix}$$

$$= \begin{pmatrix}-12 & 3\\ 6 & 15\end{pmatrix} + \begin{pmatrix}-6 & 2\\ 1 & -4\end{pmatrix} + \begin{pmatrix}-4 & 8\\ 2 & 16\end{pmatrix}$$

$$= \begin{pmatrix}-22 & 13\\ 9 & 27\end{pmatrix}$$

2

(b) $AB = \begin{pmatrix}-4 & 1\\ 2 & 5\end{pmatrix}\begin{pmatrix}6 & -2\\ -1 & 4\end{pmatrix} = \begin{pmatrix}-24-1 & 8+4\\ 12-5 & -4+20\end{pmatrix} = \begin{pmatrix}-25 & 12\\ 7 & 16\end{pmatrix}$

2

(c) $C^{-1} = \dfrac{1}{(-16-4)}\begin{pmatrix}8 & -4\\ -1 & -2\end{pmatrix} = -\dfrac{1}{20}\begin{pmatrix}8 & -4\\ -1 & -2\end{pmatrix}$

2

d) $\text{Det } D = -2\begin{vmatrix}0 & 3\\ 2 & -1\end{vmatrix} - 4\begin{vmatrix}1 & 3\\ 6 & -1\end{vmatrix} + 5\begin{vmatrix}1 & 0\\ 6 & 2\end{vmatrix}$

$$= -2(0-6) - 4(-1-18) + 5(2-0)$$

$$= 12 + 76 + 10$$

$$= 98$$

2

$\boxed{\text{Threshold} = \frac{5}{8}}$

Outcome 3

3

Q5.

$f(x) = e^{3x}$ $\qquad f(0) = e^0 = 1$

$f'(x) = 3e^{3x}$ $\qquad f'(0) = 3$

$f''(x) = 9e^{3x}$ $\qquad f''(0) = 9$

$f'''(x) = 27e^{3x}$ $\qquad f'''(0) = 27$

$$\boxed{f(x) \simeq f(0) + f'(0)\frac{x}{1!} + f''(0)\frac{x^2}{2!} + f'''(0)\frac{x^3}{3!} + \dots + f^n(0)\frac{x^n}{n!}}$$

• know MacLaurin + f(0) = 1

$$e^{3x} \simeq 1 + 3\left(\frac{x}{1!}\right) + 9\left(\frac{x^2}{2!}\right) + 27\left(\frac{x^3}{3!}\right) + \dots$$

$$= 1 + 3x + \frac{9x^2}{2} + \frac{9x^3}{2} + \dots$$

3

$\therefore$ First 3 terms are $e^{3x} = 1 + 3x + \frac{9x^2}{2} + \dots$ •

Q6. $x_{n+1} = \ln x_n + 2$, $x_0 = 3$

$x_0 = 3$

$x_1 = 3.098612$

$x_2 = 3.130954$ •

$x_3 = 3.141338$

3

$\vdots$

$x_{12} = 3.146193$

$x_{13} = 3.146193$ • $\Rightarrow$ $x = 3.15$ • to 2dps

2 iterations leading to same value.

Threshold = $\frac{4}{6}$

Outcome 4

4

Q7. $\boxed{\dfrac{dy}{dx} + \dfrac{1}{x^2}y = x^4 e^{1/x}}$

$I(x) = e^{\int P(x)}$

$I(x) = e^{\int x^{-2}}$

$= e^{\left(\frac{x^{-1}}{-1}\right)}$

$= e^{-1/x}$

or $\dfrac{1}{e^{1/x}}$ •

$$e^{-1/x}\left(\frac{dy}{dx} + \frac{1}{x^2}y\right) = e^{-1/x}\left(x^4 e^{1/x}\right)$$

$$\boxed{\frac{d}{dx}\left(I(x)\,y\right) = I(x)\,f(x)}$$

$$\frac{d}{dx}\left(e^{-1/x}y\right) = e^{-1/x}\left(x^4 e^{1/x}\right) \quad \bullet$$

State Modified Equation.

$$\int \frac{d}{dx}\left(\frac{y}{e^{1/x}}\right) \cdot dx = \int x^4\,dx \quad \bullet$$

5

$$\frac{y}{e^{1/x}} = \frac{x^5}{5} + c \quad \bullet$$

$$y = e^{1/x}\left(\frac{x^5}{5} + c\right) \quad \bullet$$

- $I(x)$
- Modified equation
- Integrating both sides $\int$
- RHS of Integration.
- Rearrange to find $y =$

(* No c ⇒ −2 Marks)

$\boxed{\text{Threshold} = \frac{3}{5}}$

Outcome 5

$$\boxed{\sum_{r=1}^{n} (3r-1) = \frac{n(3n+1)}{2}}$$

5

Let $n=1$

$$3\times1 - 1 = 2 \quad \& \quad \frac{1(3\times1+1)}{2} = \frac{4}{2} = 2$$

LHS = RHS ✓ true for $n=1$

Assume true for $n=k$

$$\sum_{r=1}^{n=k} (3r-1) = \frac{k(3k+1)}{2}$$

Consider $n=k+1$

$$\sum_{r=1}^{n=k+1} (3r-1) = \sum_{r=1}^{n=k} (3r-1) + (3(k+1)-1) = \frac{k(3k+1)}{2} + (3(k+1)-1)$$

$$= \frac{k(3k+1)}{2} + (3k+3-1)$$

$$= \frac{k(3k+1)}{2} + \frac{2(3k+2)}{2}$$

$$= \frac{k(3k+1)+2(3k+2)}{2}$$

$$= \frac{3k^2+k+6k+4}{2}$$

$$= \frac{3k^2+7k+4}{2}$$

$$= \frac{(3k+4)(k+1)}{2}$$

$$= \frac{(k+1)(3k+3+1)}{2}$$

$$= \frac{(k+1)(3(k+1)+1)}{2}$$

Statement As true for $n=1$.
and also Assumed true for $n=k$, and by Proof of Mathematical Induction true for $n=k+1$ As Required.
Statement is true $\forall n \geq 1$.

Outcome 5

(6)

Q9.

$$\left.\begin{aligned} 2516 &= 3 \cdot 646 + 578 \\ 646 &= 1 \cdot 578 + 68 \end{aligned}\right\}$$

$$\left.\begin{aligned} 578 &= 8 \cdot 68 + 34 \\ 68 &= 2 \cdot 34 + 0 \end{aligned}\right\}$$

(3)

$$\boxed{\gcd(2516, 646) = \underline{\underline{34}}}$$

$$\boxed{\text{Threshold} = 5/8}$$

2nd Set

Outcome 1

Q1. $\underline{a} = 2\underline{j} - \underline{k}$

$\underline{b} = \underline{i} - \underline{j} + \underline{k}$

$$\underline{a} \times \underline{b} = \begin{pmatrix} \underline{i} & \underline{j} & \underline{k} \\ 0 & 2 & -1 \\ 1 & -1 & 1 \end{pmatrix}$$

$$= \begin{vmatrix} 2 & -1 \\ -1 & 1 \end{vmatrix} \underline{i} - \begin{vmatrix} 0 & -1 \\ 1 & 1 \end{vmatrix} \underline{j} + \begin{vmatrix} 0 & 2 \\ 1 & -1 \end{vmatrix} \underline{k}$$

$$= (2-1)\underline{i} - (0-(-1))\underline{j} + (0-2)\underline{k}$$

$$\therefore \underline{a} \times \underline{b} = \underline{i} - \underline{j} - 2\underline{k}$$

3

Q2. $\underline{d} = \overrightarrow{AB} = \underline{b} - \underline{a} = \begin{pmatrix} 3 \\ 1 \\ 7 \end{pmatrix} - \begin{pmatrix} 1 \\ -1 \\ 4 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 3 \end{pmatrix}$

$r = a + \lambda d$ $\quad \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 4 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 2 \\ 3 \end{pmatrix}$

2

$$\therefore x = 1 + 2\lambda \;;\; y = -1 + 2\lambda \;\&\; z = 4 + 3\lambda$$

Q3. $\boxed{r.n = a.n}$ $\quad \underline{n} = 2\underline{i} - \underline{j} + \underline{k}, \; \underline{a} = 2\underline{i} + 3\underline{j} + 7\underline{k}$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ 7 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} = 4 - 3 + 7 = 8$$

2

$$\therefore 2x - y + z = 8$$

Th = 5/7

Outcome 2

Q4. a) $3A - B + 2C = 3\begin{bmatrix} 5 & 1 \\ 4 & 6 \end{bmatrix} - \begin{bmatrix} 3 & -2 \\ 1 & 5 \end{bmatrix} + 2\begin{bmatrix} 4 & 2 \\ -5 & -3 \end{bmatrix}$

$= \begin{pmatrix} 15 & 3 \\ 12 & 18 \end{pmatrix} + \begin{bmatrix} -3 & 2 \\ -1 & -5 \end{bmatrix} + \begin{bmatrix} 8 & 4 \\ -10 & -6 \end{bmatrix}$

$= \begin{bmatrix} 20 & 9 \\ 1 & 7 \end{bmatrix}$

2

(b) $AB = \begin{bmatrix} 5 & 1 \\ 4 & 6 \end{bmatrix}\begin{bmatrix} 3 & -2 \\ 1 & 5 \end{bmatrix} = \begin{bmatrix} 15+1 & -10+5 \\ 12+6 & -8+30 \end{bmatrix} = \begin{bmatrix} 16 & -5 \\ 18 & 22 \end{bmatrix}$

2

---

c) $C = \begin{bmatrix} 4 & 2 \\ -5 & -3 \end{bmatrix}$ $C^{-1} = \frac{1}{(-12-(-10))}\begin{bmatrix} -3 & -2 \\ 5 & 4 \end{bmatrix}$

$= -\frac{1}{2}\begin{bmatrix} -3 & -2 \\ 5 & 4 \end{bmatrix}$

2

---

d) $\det D = 2\begin{vmatrix} 3 & 2 \\ -4 & 1 \end{vmatrix} - 1\begin{vmatrix} 0 & 2 \\ 1 & 1 \end{vmatrix} + (-1)\begin{vmatrix} 0 & 3 \\ 1 & -4 \end{vmatrix}$

$= 2(3-(-8)) - 1(0-2) - 1(0-3)$

$= 2(11) - 1(-2) - 1(-3)$

$= 22 + 2 + 3$

$= 27$

2

Th = 5/8

Outcome 3

Q5.

$f(x) = e^{2x}$ $\quad f(0) = e^0 = 1$

$f'(x) = 2e^{2x}$ $\quad f'(0) = 2$

$f''(x) = 4e^{2x}$ $\quad f''(0) = 4$

$f'''(x) = 8e^{2x}$ $\quad f'''(0) = 8$

$$\boxed{f(x) \approx f(0) + f'(0)\frac{x}{1!} + f''(0)\frac{x^2}{2!} + f'''(0)\frac{x^3}{3!} + \dots + f^n(0)\frac{x^n}{n!}}$$

$+ f(0) = 1$

$$\therefore \quad e^{2x} = 1 + 2\frac{x}{1!} + 4\frac{x^2}{2!} + 8\frac{x^3}{3!} + \dots$$

$$= 1 + 2x + 2x^2 + \frac{4x^3}{3} + \dots$$

3

So 1st 3 terms of e^{2x} = $1 + 2x + 2x^2 + \dots$

Q6.

$$x_{n+1} = (2x_n + 1)^{1/3} \quad ; \quad x_0 = 2$$

$x_0 = 2$

$x_1 = 1.709976$

$x_2 = 1.641116$

$x_3 = 1.623890$

$x_4 = 1.619524$

$\vdots$

$x_9 = 1.6180356$

$x_{10} = 1.6180344$

$x_{11} = 1.6180341$

$x_{12} = 1.6180340$

$x_{13} = 1.6180340$

$\Rightarrow \quad x = 1.62$

3

Th = 4/6

Outcome 4

Q7. $\frac{dy}{dx} + \frac{1}{x}y = e^x$

$I(x) = e^{\int P(x)dx}$

$I(x) = e^{\int 1/x \, dx}$

$= e^{\ln x}$

$= x$

$x\left(\frac{dy}{dx} + \frac{1}{x}y\right) = x\left(e^x\right)$

$\boxed{\frac{d}{dx}(I(x)y) = I(x)f(x)}$

$\frac{d}{dx}(xy) = xe^x$

$\int \frac{d}{dx}(xy).dx = \int xe^x.dx$

Int by Parts!!!

let $u = x$ $\quad v = e^x$

$u' = 1$ $\quad v' = e^x$

$\boxed{\int uv' = uv - \int u'v}$

$xy = \{xe^x - \int e^x dx\}$

$xy = xe^x - e^x + c$

$y = e^x - \frac{e^x}{x} + \frac{c}{x}$

5

or other variations

$y = \frac{xe^x - e^x + c}{x}$

$y = \frac{e^x(x-1) + c}{x}$ etc.

Th = 3/5

Outcome 5

$$\boxed{\sum_{k=1}^{n} k = \frac{1}{2}n(n+1)}$$

[* As k used, either use a different letter or 'K']

Let $n=1$

$1 = \frac{1}{2}(1)(1+1)$

$1 = \frac{1}{2} \times 2 = 1$ ✓ true for $n=1$

Assume true for $n=K$

$$\sum_{k=1}^{n=K} k = \frac{1}{2}K(K+1)$$

Consider $n=K+1$

$$\sum_{k=1}^{n=K+1} k = \sum_{k=1}^{n=K} k \boxed{+(K+1)} = \frac{1}{2}K(K+1)\boxed{+(K+1)}$$

$$= \frac{1}{2}K(K+1) + \frac{2(K+1)}{2}$$

$$= \frac{(K+1)}{2}[K+2]$$

$$= \frac{(K+1)((K+1)+1)}{2}$$

If $N=(K+1)$ this therefore becomes →

$$= \frac{N(N+1)}{2}$$

as required

Statement As true for $n=1$, assumed true for $n=K$ and by Proof of Mathematical Induction also true for $n=(K+1)$. Statement is true for $\forall n \in \mathbb{N}$.

5

Outcome 5

Q9.

$$1147 = 1 \cdot 851 + 296$$
$$851 = 2 \cdot 296 + 259$$
$$296 = 1 \cdot 259 + 37$$
$$259 = 7 \cdot \underline{\underline{37}} + 0$$

3

$\therefore \gcd(1147, 851) = \underline{\underline{37}}$

Th = 5/8