AH 2010

Q1. a) $f(x) = e^x . \sin(x^2)$

$u = e^x \quad v = \sin(x^2)$

$u' = e^x \quad v' = \cos(x^2) . 2x$

$f'(x) = u'v + uv'$

$= e^x . \sin(x^2) + e^x . [2x \cos(x^2)]$

$= e^x [\sin(x^2) + 2x \cos(x^2)]$

3

1b). $g(x) = \dfrac{x^3}{(1 + \tan x)}$

$u = x^3 \quad v = 1 + \tan x$

$u' = 3x^2 \quad v' = \sec^2 x$

$g'(x) = \dfrac{u'v - uv'}{v^2}$

$= \dfrac{3x^2(1 + \tan x) - x^3 \sec^2 x}{(1 + \tan x)^2}$

3

$= \dfrac{3x^2 + 3x^2 \tan x - x^3 \sec^2 x}{(1 + \tan x)^2}$

$= \dfrac{x^2 [3(1 + \tan x) - x \sec^2 x]}{(1 + \tan x)^2}$

Not much else you can do to simplify this

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Q2. $\boxed{\text{Geometric} \Rightarrow a, ar, ar^2, \ldots, ar^{n-1}}$

ie. $u_n = ar^{n-1}$

Then $u_2 = ar^1 = -6$

& $u_3 = ar^2 = 3$

$$\frac{u_3}{u_2} \Rightarrow \frac{ar^2}{ar^1} = \frac{3}{-6}$$

5

$$\therefore \underline{r = -\tfrac{1}{2}}$$

If $ar = -6$ & $r = -\tfrac{1}{2}$

$$ar = -6$$
$$a \times \frac{-1}{2} = -6$$
$$\therefore \underline{a = 12}$$

As $-1 < -\tfrac{1}{2} < 1$ or $-1 < r < 1$
the sequence tends to a limit

$$S_\infty = \frac{a}{1-r} = \frac{12}{1-(-\frac{1}{2})} = \frac{12}{3/2} = \frac{24}{3} = \underline{\underline{8}}$$

$$\therefore \underline{S_\infty = 8}$$

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Q3. a) $\displaystyle\int \frac{x^3}{1+x^8}\,dx$

Let $t = x^4$

$\dfrac{dt}{dx} = 4x^3$

$\therefore \quad dx = \dfrac{dt}{4x^3}$

& $1 + x^8 = 1 + (x^4)^2$
$= 1 + t^2$

$= \displaystyle\int \frac{x^3}{1+(x^4)^2}\,dx$

$= \displaystyle\int \frac{x^3}{1+(t)^2} \cdot \frac{dt}{4x^3}$

$= \dfrac{1}{4}\displaystyle\int \frac{dt}{1+t^2}$

$= \dfrac{1}{4}\tan^{-1}\left(\dfrac{t}{1}\right) + C$ 3

$= \frac{1}{4}\tan^{-1}(x^4) + C$

b) $\displaystyle\int x^2 \ln x\,dx \Rightarrow$ Integ by Parts

$u = x^3/3 \quad v = \ln x$
$u' = x^2 \quad v' = 1/x$

$\boxed{\int u'v = uv - \int uv' \text{ [Must diff } \ln|x|\text{]}}$

$= \dfrac{x^3}{3}\ln|x| - \displaystyle\int\left(\frac{x^3}{3} \times \frac{1}{x}\right)dx$ 4

$= \dfrac{x^3}{3}\ln|x| - \dfrac{1}{3}\displaystyle\int x^2\,dx$

$= \dfrac{x^3}{3}\ln|x| - \dfrac{1}{3}\left(\dfrac{x^3}{3}\right) + C \Rightarrow \dfrac{x^3}{9}\left[3\ln|x| - 1\right] + C$

Q4. Scale factor of 2 $\Rightarrow \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ •

CLOCKWISE rotation of 60° $\Rightarrow \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$

$\Rightarrow$ Use $(-60°)$ or $(300°)$

Let $\theta = -60°$ here $\rightarrow$

$$\begin{pmatrix} \cos(-60) & -\sin(-60) \\ \sin(-60) & \cos(-60) \end{pmatrix}$$

$$= \begin{pmatrix} \cos 60 & +\sin 60 \\ -\sin 60 & \cos 60 \end{pmatrix}$$

$$= \begin{pmatrix} 1/2 & \sqrt{3}/2 \\ -\sqrt{3}/2 & 1/2 \end{pmatrix}$$ •

Then $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & 1/2 \end{pmatrix}$ •

$$= \begin{pmatrix} (2\times\frac{1}{2}+0) & (2\times\frac{\sqrt{3}}{2}+0) \\ (0-2\times\frac{\sqrt{3}}{2}) & (0+2\times 1/2) \end{pmatrix}$$ (4)

$$= \begin{pmatrix} 1 & \sqrt{3} \\ -\sqrt{3} & 1 \end{pmatrix}$$ •

Q5. $\binom{n+1}{3} - \binom{n}{3}$

$$= \frac{(n+1)!}{3!((n+1)-3)!} - \frac{n!}{3!(n-3)!}$$

$$= \frac{(n+1)n!}{3!(n-2)!} - \frac{n!}{3!(n-3)!}$$

$$= \frac{n!(n+1)}{3!(n-2)(n-3)!} - \frac{n!}{3!(n-3)!}$$

$$= \frac{n!}{3!(n-3)!}\left[\frac{(n+1)}{(n-2)} - 1\right]$$

$$= \frac{n!}{3!(n-3)!}\left[\frac{n+1-(n-2)}{(n-2)}\right]$$

$$= \frac{n!}{3!(n-3)!}\left[\frac{3}{(n-2)}\right]$$

4

$$= \frac{n!}{2!(n-2)(n-3)!} = \frac{n!}{2!(n-2)!} = \binom{n}{2}$$

as required

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Q6. $\underline{u} = -2\underline{i} + 5\underline{k}$, $v = 3\underline{i} + 2\underline{j} - \underline{k}$ & $w = -\underline{i} + \underline{j} + 4\underline{k}$

Method 1 $\quad \underline{u}.(\underline{v} \times \underline{w})$

$$= \begin{pmatrix} -2 & 0 & 5 \\ 3 & 2 & -1 \\ -1 & 1 & 4 \end{pmatrix}$$

$$= -2\begin{vmatrix} 2 & -1 \\ 1 & 4 \end{vmatrix} - 0 + 5\begin{vmatrix} 3 & 2 \\ -1 & 1 \end{vmatrix}$$

$$= -2(8-(-1)) + 5(3-(-2))$$
$$= -2(9) + 5(5)$$
$$= -18 + 25$$
$$= \underline{\underline{7}}$$

4

Method 2 $\quad \underline{v} \times \underline{w} = \begin{pmatrix} \underline{i} & \underline{j} & \underline{k} \\ 3 & 2 & -1 \\ -1 & 1 & 4 \end{pmatrix}$

$$= \underline{i}\begin{vmatrix} 2 & -1 \\ 1 & 4 \end{vmatrix} - \underline{j}\begin{vmatrix} 3 & -1 \\ -1 & 4 \end{vmatrix} + \underline{k}\begin{vmatrix} 3 & 2 \\ -1 & 1 \end{vmatrix}$$

$$= \underline{i}(8-(-1)) - \underline{j}(12-1) + \underline{k}(3-(-2))$$

$$= 9\underline{i} - 11\underline{j} + 5\underline{k}$$

$$\underline{u}.(v \times w) = \begin{pmatrix} -2 \\ 0 \\ 5 \end{pmatrix} \cdot \begin{pmatrix} 9 \\ -11 \\ 5 \end{pmatrix} = -18 + 0 + 25 = \underline{\underline{7}}$$

Q7. $\int_1^2 \frac{3x+5}{(x+1)(x+2)(x+3)}\,dx = \int_1^2 \left(\frac{A}{(x+1)} + \frac{B}{(x+2)} + \frac{C}{(x+3)}\right)dx$

$\therefore \quad 3x+5 = A(x+2)(x+3) + B(x+1)(x+3) + C(x+1)(x+2)$

Let $x = -1$ $\quad -3+5 = A(-1+2)(-1+3) + 0 + 0$

$2 = A(1)(2)$

$2A = 2 \Rightarrow A = 1$

Let $x = -2$ $\quad -6+5 = 0 + B(-2+1)(-2+3) + 0$

$-1 = B(-1)(1)$

$-B = -1 \Rightarrow B = 1$

Let $x = -3$ $\quad -9+5 = 0 + 0 + C(-3+1)(-3+2)$

$-4 = C(-2)(-1)$

$2C = -4 \Rightarrow C = -2$

$\therefore \int_1^2 \frac{3x+5\quad dx}{(x+1)(x+2)(x+3)} = \int_1^2 \left(\frac{1}{(x+1)} + \frac{1}{(x+2)} - \frac{2}{(x+3)}\right)dx$

$= \Big[\ln|x+1| + \ln|x+2| - 2\ln|x+3|\Big]_1^2$

$= \left[\ln\left|\frac{(x+1)(x+2)}{(x+3)^2}\right|\right]_1^2$

6

$= \ln\left|\frac{3\times4}{5^2}\right| - \ln\left|\frac{2\times3}{4^2}\right|$

$= \ln\left|\frac{12}{25} \div \frac{6}{16}\right|$

$= \ln\left|\frac{\cancel{12}^{2}}{25} \times \frac{16}{\cancel{6}_1}\right| = \ln\left|\frac{32}{25}\right|$

Q8 a) Let $2m+1$ and $2n+3$ be 2 odd integers independent of one another with n & m being positive integers also.

Then Product of 2 Odd Integers:

$$
\begin{aligned}
(2m+1)(2n+3) &= 4mn + 6m + 2n + 3 \\
&= 2(n + 3m + 2mn) + 3 \\
&= 2(k) + 3
\end{aligned}
$$

(where $k = n + 3m + 2mn$)

As $2k$ is even for all positive integers k Then $2k+3$ is always odd. 2

Thus the product of 2 odd integers is ODD

b) If p is an odd integer let $p = 2n+1$, where n is a positive integer. 4

Let $n=1$ $p^1 = 2+1 = 3$ which is odd. true for $n=1$

Assume true for $n=k$ then $p^k = 2k+1$

Consider $n=k+1$ $p^{k+1} = p^1 . p^k = (2+1)(2k+1)$
$= 3(2k+1)$
$=$ Odd x Odd
$\Rightarrow$ Always Odd.

Thus by proof by Induction as true for $n=1$, assumed true for $n=k$ and by mathematical induction true for $n=k+1$ Assume true for all $n \in N$.

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Q9. $f(x) = (1+\sin^2 x) = (1+(\sin x)^2)$ $\quad f(0) = 1+0 = \underline{1}$

$f'(x) = 2(\sin x)^1 . \cos x$ $\quad f'(0) = \sin 0 = \underline{0}$

$= \sin 2x$

$f''(x) = 2\cos 2x$ $\quad f''(0) = 2\cos 0 = \underline{2}$

$f'''(x) = -4\sin 2x$ $\quad f'''(0) = -4\sin 0 = \underline{0}$

$f^{iv}(x) = -8\cos 2x$ $\quad f^{iv}(0) = -8\cos 0 = \underline{-8}$

$f^{v}(x) = +16\sin 2x$ $\quad f^{v}(0) = 16\sin 0 = \underline{0}$

$f^{vi}(x) = 32\cos 2x$ $\quad f^{vi}(0) = 32\cos 0 = \underline{32}$

$$\boxed{f(0) + f'(0)\frac{x}{1!} + f''(0)\frac{x^2}{2!} + f'''(0)\frac{x^3}{3!} + \ldots + f^n(0)\frac{x^n}{n!}}$$

$$\therefore \quad 1+\sin^2 x = (1) + (0)\frac{x}{1!} + (2)\frac{x^2}{2!} + (0)\frac{x^3}{3!} + (-8)\frac{x^4}{4!} + \ldots$$

$$= 1 + \frac{2x^2}{2} - \frac{8x^4}{24} + \ldots$$

4

$$1+\sin^2 x = 1 + x^2 - \frac{x^4}{3} + \ldots$$

are the first 3 Non-Zero terms.

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Q10. The graph is not symmetrical about the y-axis [$f(-x) \neq f(x)$] $\Rightarrow$ NOT EVEN •

The graph does not have ½-turn rotational symmetry about the origin $\Rightarrow$ NOT ODD •
[$f(-x) \neq -f(x)$]

$\therefore$ The function is NEITHER EVEN/ODD •

Q11. $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 5y = 0$

Auxillary Eqn $m^2 + 4m + 5 = 0$ •

$a = 1$, $b = 4$, $c = 5$

$$m = \frac{-4 \pm \sqrt{4^2 - 4\times1\times5}}{2\times1}$$

$$= \frac{-4 \pm \sqrt{16-20}}{2}$$

$$= \frac{-4 \pm \sqrt{-4}}{2}$$

$$= \frac{-4 \pm 2i}{2}$$

4

$\therefore m = -2 \pm i$ •

$y = e^{-2x}(A\cos x + B\sin x)$ •

is General Solution

Q11 (Ctd/..)

$$y = e^{-2x}(A\cos x + B\sin x)$$

$\left.\begin{array}{l} y = 3 \\ x = 0 \end{array}\right]$ $3 = e^0(A\cos 0 + B\sin 0)$

$3 = 1(A + 0)$

$\therefore \underline{\underline{A = 3}}$ •

$\left.\begin{array}{l} y = e^{-\pi} \\ x = \frac{\pi}{2} \end{array}\right]$ $e^{-\pi} = e^{-2\left(\frac{\pi}{2}\right)}\left(A\cos\left(\frac{\pi}{2}\right) + B\sin\left(\frac{\pi}{2}\right)\right)$

$e^{-\pi} = e^{-\pi}(A \times 0 + B \times 1)$

$1 = 1(B)$

$\therefore \underline{\underline{B = 1}}$ •

Thus Particular Solution is (3)

$$y = e^{-2x}(3\cos x + \sin x)$$ •

Q12.

If x is an irrational number we will assume $2 + x$ is rational.

Then Let $2 + x = \frac{p}{q}$ where p & q are positive integers with no common factors.

Then $x = \frac{p}{q} - 2$

$$= \frac{p - 2q}{q}$$

As p & q are positive integers

$\frac{p - 2q}{q}$ is rational

$\Rightarrow$ x is rational.

However the initial statement stated x is an irrational number $\Rightarrow$ CONTRADICTION!.

Thus by proof of Contradiction if x is an irrational number then $2 + x$ is also irrational.

4

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Q13.

$y = t^3 - \frac{5}{2}t^2$ | $x = \sqrt{t} = t^{1/2}$

$\frac{dy}{dt} = 3t^2 - 5t$ | $\frac{dx}{dt} = \frac{1}{2}t^{-1/2} = \frac{1}{2\sqrt{t}}$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{3t^2-5t}{\left(\frac{1}{2\sqrt{t}}\right)} = 2\sqrt{t} \times t(3t-5)$$

$$= 2\sqrt{t^3}\,(3t-5)$$

4

$$\frac{d^2y}{dx^2} = \frac{d}{dt}\left(\frac{dy}{dx}\right) \times \frac{dt}{dx} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}}$$

$\frac{d}{dt}\left(\frac{dy}{dx}\right) = u'v + uv'$

$u = 2t^{3/2}$ $\quad v = 3t-5$

$u' = 3t^{1/2}$ $\quad v' = 3$

$= 3\sqrt{t}$

$$= 3\sqrt{t}(3t-5) + 6\sqrt{t^3}$$

$$= 3\sqrt{t}\left[(3t-5) + 2t\right]$$

$$= 3\sqrt{t}\left[5t-5\right]$$

3

$$= 15\sqrt{t}\,(t-1)$$

$$\frac{d^2y}{dx^2} = \frac{15\sqrt{t}\,(t-1)}{\left(\frac{1}{2\sqrt{t}}\right)} = 30t(t-1) \Rightarrow 30t^2 - 30t$$

ie. $a = 30$ & $b = -30$

Q13 (Cont...) $\frac{dy}{dx} = 2\sqrt{t^3}(3t-5)$ & $\frac{d^2y}{dx^2} = 30t(t-1)$

$$\boxed{\text{Point of Inflexion when } \frac{d^2y}{dx^2} = 0}$$

$$\frac{d^2y}{dx^2} = 30t(t-1) = 0$$

$30t = 0$ $\quad$ $t-1=0$

$t=0$ $\quad$ $t=1$

But $t>0$ $\therefore$ $t=1$

$x = \sqrt{t} = \sqrt{1} = 1$

$y = t^3 - \frac{5}{2}t^2 = 1 - \frac{5}{2} = -\frac{3}{2}$ $\quad$ $\therefore$ Pt $(1, -\frac{3}{2})$ •

Gradient at $t=1$, $\frac{dy}{dx} = 2\sqrt{t^3}(3t-5)$

$= 2 \times 1 \times (3-5)$

$= 2(-2)$

$\therefore$ $m = -4$ •

$y - b = m(x-a)$

$\left(y + \frac{3}{2}\right) = -4(x-1)$

$y + \frac{3}{2} = -4x + 4$ •

3

$2y + 3 = -8x + 8$

$\therefore$ $8x + 2y - 5 = 0$ (or alternative rearrangement

AH 2010

Q14.

$$\begin{aligned} x - y + z &= 1 \\ x + y + 2z &= 0 \\ 2x - y + az &= 2 \end{aligned} \rightarrow \left(\begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ 1 & 1 & 2 & 0 \\ 2 & -1 & a & 2 \end{array}\right) \begin{array}{c} r_1 \\ r_2 \\ r_3 \end{array}$$

$$\begin{array}{c} \\ r_2 - r_1 \\ r_3 - 2r_1 \end{array} \left(\begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ 0 & 2 & 1 & -1 \\ 0 & 1 & a-2 & 0 \end{array}\right) \begin{array}{c} r_1 \\ r_2 \\ r_3 \end{array} \rightarrow \begin{array}{c} \\ \\ 2r_3 \end{array} \left(\begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ 0 & 2 & 1 & -1 \\ 0 & 2 & 2a-4 & 0 \end{array}\right) \begin{array}{c} r_1 \\ r_2 \\ r_3 \end{array}$$

$$\begin{array}{c} \\ \\ r_3 - r_2 \end{array} \left(\begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ 0 & 2 & 1 & -1 \\ 0 & 0 & 2a-5 & 1 \end{array}\right) \begin{array}{c} r_1 \\ r_2 \\ r_3 \end{array}$$

5

$\therefore$ Unique solution when $2a - 5 \neq 0$

$2a \neq 5$

$a \neq \frac{5}{2}$

Then unique solutions would be :

$(2a-5)z = 1$; $2y + z = -1$; $x - y + z = 1$

$z = \dfrac{1}{(2a-5)}$

$2y = -1 - z$

$2y = -1 - \dfrac{1}{(2a-5)}$

$y = \dfrac{1}{2}\left(\dfrac{-2a+5-1}{(2a-5)}\right)$

$= \dfrac{4-2a}{2(2a-5)}$

$= \dfrac{2-a}{2a-5}$

$x = 1 + y - z$

$= 1 + \dfrac{2-a-1}{2a-5}$

$= 1 + \dfrac{1-a}{2a-5}$

or $\dfrac{2a-5+1-a}{2a-5}$

$\therefore x = \dfrac{a-4}{2a-5}$

AH 2010 (16)

Q14 (Ctd/...)

When $a = 2\cdot5$:

$$2a - 5$$
$$= 2(2\cdot5) - 5$$
$$= 5 - 5$$
$$= 0$$

i.e. $0x + 0y + 0z = 1$

$\Rightarrow$ INCONSISTENT

$\therefore$ NO SOLUTION EXISTS (1)

When $a = 3$

$$(2a-5)z = 1$$
$$(6-5)z = 1$$
$$z = 1$$

If $z = 1$

$$2y + z = -1$$
$$2y + 1 = -1$$
$$2y = -2$$
$$\therefore\ y = -1$$

&

$$x - y + z = 1$$
$$x - (-1) + 1 = 1$$
$$x + 2 = 1$$
$$x = -1$$

$\therefore$ Solution is $(-1, -1, 1)$ (1)

$$AB = \begin{pmatrix} 5 & 2 & -3 \\ 1 & 1 & -1 \\ -3 & -1 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} = \begin{pmatrix} 5-6 \\ 1-2 \\ -3+4 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}$$ (1)

$$AC = \begin{pmatrix} 5 & 2 & -3 \\ 1 & 1 & -1 \\ -3 & -1 & 2 \end{pmatrix}\begin{pmatrix} 1 & -1 & 1 \\ 1 & 1 & 2 \\ 2 & -1 & 3 \end{pmatrix} = \begin{pmatrix} 5+2-6 & -5+2+3 & 5+4-9 \\ 1+1-2 & -1+1+1 & 1+2-3 \\ -3-1+4 & 3-1-2 & -3-2+6 \end{pmatrix}$$

$$\therefore\ AC = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I$$ (2)

$\therefore$ A is the inverse of C + vice-versa.

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17

Q15. $y = x^2 \Rightarrow y^2 = x^4$ & $y^2 = 8x$

Intersect when $y^2 = y^2$:- $x^4 = 8x$

$x^4 - 8x = 0$

$x(x^3 - 8) = 0$

$\underline{\underline{x = 0}}$ $\quad x^3 = 8$

$\underline{\underline{x = 2}}$ •

$\boxed{\therefore \text{ Area of Design } = 4\int_0^2 (\text{Above} - \text{Below})\,dx}$

$\boxed{\text{Above } y^2 = 8x \;\; * \Rightarrow y = \sqrt{8x}}$

$$= 4\int_0^2 \left(\sqrt{8x} - x^2\right) dx \quad \bullet$$

•

$$= 4\left[\frac{\sqrt{8}\,x^{3/2}}{3/2} - \frac{x^3}{3}\right]_0^2 \quad \bullet$$

5

$$= 4\left[\frac{2\sqrt{8}\,\sqrt{x^3}}{3} - \frac{x^3}{3}\right]_0^2$$

$$= \frac{4}{3}\left[\left[2\sqrt{8}\times(\sqrt{2})^3 - (2)^3\right] - [0]\right]$$

$$= \frac{4}{3}\left[2\times\sqrt{8}\times\sqrt{8} - 8 - 0\right]$$

$$= \frac{4}{3}\left[16 - 8\right] \quad \bullet$$

$$= \underline{\underline{\frac{32}{3}}} \text{ or } \underline{10\tfrac{2}{3}} \text{ units}^2$$

Q15.
(CH/..)

Volume of Revolution on y-axis ⇒ need to use dy not dx so in terms of y

$\therefore \boxed{\text{Volume} = \int_a^b \pi x^2 \, dy}$

$y = x^2 \Rightarrow x^2 = y$ If $x = 0 : y = 0$

$y^2 = 8x \Rightarrow y^4 = 64x^2$ $x = 2 \quad y = 2^2 = 4$

$\therefore x^2 = \frac{y^4}{64}$

$$\text{Volume} = \int_0^4 \pi \left(y - \frac{y^4}{64}\right) dy$$

$$= \pi \int_0^4 \left(y - \frac{y^4}{64}\right) dy$$

5

$$= \pi \left[\frac{y^2}{2} - \frac{y^5}{320} \right]_0^4$$

$$= \pi \left[\left(\frac{16}{2} - \frac{16 \times 64}{320} \right) - (0) \right]$$

$$= \pi \left[8 - \frac{16 \times 2}{10} \right]$$

$$= \pi \left[8 - 3\tfrac{1}{5} \right]$$

$$= \pi \left[4\tfrac{4}{5} \right]$$

$$= \frac{24\pi}{5} \text{ units}^3$$

AH 2010

Q16. $z^3 = (r(\cos\theta + i\sin\theta))^3$
$= r^3(\cos(3\theta) + i\sin(3\theta))$ (1)

$\left(\cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right)\right)^3 = \cos\left(3\times\frac{2\pi}{3}\right) + i\sin\left(3\times\frac{2\pi}{3}\right)$
$= \cos(2\pi) + i\sin(2\pi)$
$= 1 + 0i$
$\therefore a = 1$ & $b = 0$ (2)

$\boxed{\begin{array}{l} z^3 = 8 \\ \quad = 8 \times 1 \\ \therefore z^3 = 8(\cos(2\pi) + i\sin(2\pi)) \end{array}}$

As $z^3 \Rightarrow$ 3 roots exist by finding $(z^3)^{1/3}$

$\Rightarrow z_1 = 8^{1/3}(\cos(2\pi) + i\sin(2\pi))^{1/3}$
$= 2\left(\cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right)\right)$
$= 2\left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right) \Rightarrow z_1 = -1 + i\sqrt{3}$

$z_2 = 8^{1/3}(\cos(2\pi + 2\pi) + i\sin(2\pi + 2\pi))^{1/3}$
$= 2\left(\cos\left(\frac{4\pi}{3}\right) + i\sin\left(\frac{4\pi}{3}\right)\right)$
$= 2\left(-\frac{1}{2} - i\frac{\sqrt{3}}{2}\right) \Rightarrow z_2 = -1 - i\sqrt{3}$

$z_3 = 8^{1/3}(\cos(6\pi) + i\sin(6\pi))^{1/3}$
$= 2(\cos(6\pi/3) + i\sin(6\pi/3))$
$= 2(\cos(2\pi) + i\sin(2\pi))$
$= 2(1 + 0i)$
$= 2$ $\Rightarrow z_3 = 2$ (4)

Q16. (Ctd...)

Alternative method to find 3 roots

$z^3 = 8$

$\therefore z^3 - 8 = 0$

$$\begin{array}{c|cccc} & 1 & 0 & 0 & -8 \\ 2 & \downarrow & 2 & 4 & 8 \\ \hline & 1 & 2 & 4 & \underline{0} \end{array}$$

$\therefore z^3 - 8 = 0$

$(z-2)(z^2+2z+4) = 0$

$(z-2)(z+1-\sqrt{3}i)(z+1+\sqrt{3}i)=0$

$z_1 = 2$; $z_2 = -1+\sqrt{3}i$; $z_3 = -1-\sqrt{3}i$

are 3 roots of $z^3 = 8$

$$z = \frac{-2 \pm \sqrt{2^2 - 4\times 1\times 4}}{2\times 1}$$
$$= \frac{-2 \pm \sqrt{4-16}}{2}$$
$$= \frac{-2 \pm \sqrt{12}\, i}{2}$$
$$= \frac{-2 \pm \sqrt{4}\sqrt{3}\, i}{2}$$
$$= \frac{-2 \pm 2\sqrt{3}\, i}{2}$$
$$= \underline{\underline{-1 \pm \sqrt{3}\, i}}$$

a) $\therefore z_1 + z_2 + z_3$

$= 2 + (-1+\sqrt{3}i) + (-1-\sqrt{3}i)$

$= \underline{\underline{0}}$

b) $z^3 = 8 \quad \therefore z_1^3 = z_2^3 = z_3^3 = 8$

$z^6 = (z^3)^2 \quad \therefore z_1^6 + z_2^6 + z_3^6$

$= (z_1^3)^2 + (z_2^3)^2 + (z_3^3)^2$

$= 8^2 + 8^2 + 8^2$

$= 3 \times 64$

$= \underline{\underline{192}}$

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