SECTION A (Mathematics 1 and 2) 2001

Solutions

A1.

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 10 \\ 2 & -1 & 3 & 4 \\ 1 & 0 & 2 & 20 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 10 \\ 0 & -3 & 1 & -16 \\ 0 & -1 & 1 & 10 \end{array}\right) \quad \begin{array}{l} \\ r_2 - 2r_1 \\ r_3 - r_1 \end{array}$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 10 \\ 0 & -3 & 1 & -16 \\ 0 & 0 & \tfrac{2}{3} & 15\tfrac{1}{3} \end{array}\right) \quad \begin{array}{l} \\ \\ r_3' - \tfrac{1}{3}r_2' \end{array}$$

$$z = 23;\ y = 13;\ x = -26$$

A2. (a)

$$f(x) = (2 + x)\tan^{-1}\sqrt{x - 1}$$

$$f'(x) = \tan^{-1}\sqrt{x-1} + \frac{(2+x)\frac{1}{2}(x-1)^{-1/2}}{1 + (x-1)}$$

$$= \tan^{-1}\sqrt{x - 1} + \frac{2 + x}{2x\sqrt{x - 1}}$$

(b)

$$g(x) = e^{\cot 2x}$$

$$g'(x) = -2\operatorname{cosec}^2 2x\, e^{\cot 2x}$$

A 3.

$$\int_0^{\pi/4} 2x\sin 4x\,dx = \left[2x\int \sin 4x\,dx - \int\left(2\int \sin 4x\,dx\right)dx\right]_0^{\pi/4}$$

$$= \left[2x\,\frac{1}{4}(-\cos 4x) + \frac{1}{2}\int \cos 4x\,dx\right]_0^{\pi/4}$$

$$= \left[-\frac{1}{2}x\cos 4x + \frac{1}{8}\sin 4x\right]_0^{\pi/4}$$

$$= -\frac{1}{2}\frac{\pi}{4}(-1) = \frac{\pi}{8}$$

A4. When $n = 1$, LHS $= 2$,
RHS $= \frac{1}{2} \times 1 \times 4 = 2$, thus true for $n = 1$.
Assume true for $n = k$ and consider the case when $n = k + 1$

$$2 + 5 + 8 + \ldots + (3k - 1) + \big(3(k + 1) - 1\big)$$

$$= \tfrac{1}{2}k(3k + 1) + 3k + 2$$

$$= \tfrac{1}{2}\left(3k^2 + 7k + 4\right)$$

$$= \tfrac{1}{2}(k + 1)(3k + 4)$$

$$= \tfrac{1}{2}(k + 1)\big(3(k + 1) + 1\big)$$

i.e. since true for $n = k$ implies true for $n = k + 1$ and true for $n = 1$, the result is true for all $n \geqslant 1$.

A5. (a)

$$\frac{x}{x^2 - 1} = \frac{A}{x - 1} + \frac{B}{x + 1}$$
$$= \frac{\frac{1}{2}}{x - 1} + \frac{\frac{1}{2}}{x + 1}$$

(b)

$$\frac{x^3}{x^2 - 1} = x + \frac{x}{x^2 - 1}$$
$$= x + \frac{\frac{1}{2}}{x - 1} + \frac{\frac{1}{2}}{x + 1}.$$
$$\int \frac{x^3}{x^2-1}\,dx = \int x + \frac{\frac{1}{2}}{x - 1} + \frac{\frac{1}{2}}{x+1}\,dx$$
$$= \tfrac{1}{2}x^2 + \tfrac{1}{2}\ln(x-1) + \tfrac{1}{2}\ln(x+1) + c$$
$$= \tfrac{1}{2}\left(x^2 + \ln(x^2 - 1)\right) + c$$

A6.

$$\left(x^2 - \frac{2}{x}\right)^4 = x^8 - 4x^6.\frac{2}{x} + 6x^4.\frac{4}{x^2} - 4x^2\frac{8}{x^3} + \frac{16}{x^4}$$
$$= x^8 - 8x^5 + 24x^2 - \frac{32}{x} + \frac{16}{x^4}$$

A7. (a)

$$xy + y^2 = 2$$
$$x\frac{dy}{dx} + y + 2y\frac{dy}{dx} = 0$$
$$(x + 2y)\frac{dy}{dx} = -y$$
$$\frac{dy}{dx} = \frac{-y}{x + 2y}$$

(b) When $x = 1$ and $y = 1$, $\dfrac{dy}{dx} = \dfrac{-1}{1 + 2} = -\dfrac{1}{3}$

Equation is $(y - 1) = -\frac{1}{3}(x - 1)$.

A 8. (a)

$$f(x) = \frac{x^2 + 6x + 12}{x + 2} = x + 4 + \frac{4}{x + 2}$$

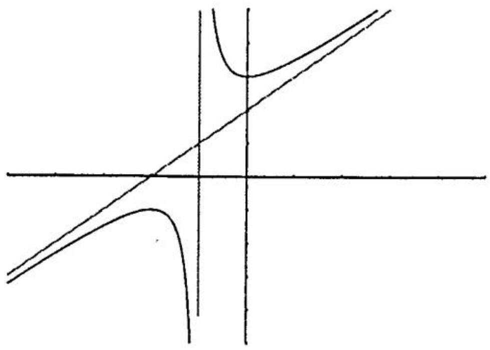
(b) Vertical asymptote $x = -2$. Slant asymptote $y = x + 4$

(c)

$$f'(x) = 1 - \frac{4}{(x + 2)^2} = 0 \text{ at S.V.}$$
$$(x + 2)^2 = 4$$
$$x + 2 = \pm 2$$
$$x = 0 \text{ or } x = -4.$$
$$f''(x) = \frac{8}{(x + 2)^3}$$

When $x = 0, f(0) = 6$ and $f''(0) > 0$ so $(0, 6)$ is a minimum turning point.
When $x = -4, f(-4) = -2$ and $f''(-4) < 0$ so $(-4, -2)$ is a maximum turning point.

(d)



(e)

$$-2 < k < 6$$

A9. (a)

$$-1 = \cos\pi + i\sin\pi$$

(b) Let $z = \cos\theta + i\sin\theta$, then $z^3 = \cos 3\theta + i\sin 3\theta$

$$\cos 3\theta = -1 \Rightarrow 3\theta = \pi \text{ or } 3\pi \text{ or } 5\pi$$

$$\Rightarrow \theta = \frac{\pi}{3} \text{ or } \pi \text{ or } \frac{5\pi}{3}$$

The roots are $\cos\frac{\pi}{3} + i\sin\frac{\pi}{3} = \frac{1}{2}(1 + i\sqrt{3})$, $\cos 3\pi + i\sin 3\pi = -1$ and

$$\cos\frac{5\pi}{3} + i\sin\frac{5\pi}{3} = \frac{1}{2}(1 - i\sqrt{3})$$

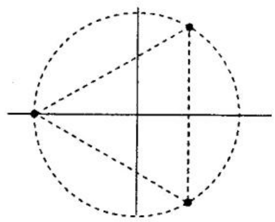
$$\left[\frac{1}{2}(1 + i\sqrt{3})^2\right] = \frac{1}{4}\left(1 + 2i\sqrt{3} - 3\right)$$

$$= -\frac{1}{2}(1 - i\sqrt{3})$$

$$\left[\frac{1}{2}(1 - i\sqrt{3})^2\right] = \frac{1}{4}\left(1 - 2i\sqrt{3} - 3\right)$$

$$= -\frac{1}{2}(1 + i\sqrt{3})$$

(c)



Acceptable comments are:
The points are on the unit circle and are equally spaced.
The solutions form an equilateral triangle.

A10 (a)

$$\frac{dM}{dt} = kM$$

$$\int \frac{dM}{M} = \int k\,dt$$

$$\ln M = kt + c$$

$$t = 0,\ M = M_0 \Rightarrow c = \ln M_0$$

$$M = M_0\,e^{kt}$$

(b) When $t = 30, M = \frac{1}{2}M_0$ so

$$e^{30k} = 0{\cdot}5$$

$$k = \frac{1}{30}\ln 0{\cdot}5 \approx -0{\cdot}0231$$

(c) When $t = 35$

$$\frac{M}{M_0} = e^{35k} \approx 0{\cdot}4454 \approx 45\%$$

(d) When $\frac{M}{M_0} = 0{\cdot}25$

$$e^{kt} = 0{\cdot}25$$

$$t = \frac{1}{k}\ln 0{\cdot}25 = \frac{30\ \ln 0{\cdot}25}{\ln 0{\cdot}5} = 60$$

The manufacturer is justified.

Advanced Higher Mathematics 2001
SECTION B (Mathematics 3)

B1.

$$\begin{aligned} 149 &= 1 \times 139 + 10 \\ 139 &= 13 \times 10 + 9 \\ 10 &= 1 \times 9 + 1 \\ 1 &= 10 - 9 \\ &= 10 - (139 - 13 \times 10) \\ &= 14 \times (149 - 139) - 139 \\ &= 14 \times 149 - 15 \times 139 \end{aligned}$$

i.e. $x = 14$ and $y = -15$.

B2.

$$\frac{dy}{dx} + \frac{y}{x} = x$$

Integrating factor is $e^{\int \frac{1}{x}dx} = e^{\ln x} = x$

$$\therefore x\frac{dy}{dx} + y = x^2$$

$$\frac{d}{dx}(xy) = x^2$$

Hence $xy = \int x^2 dx = \frac{1}{3}x^3 + c$

Thus $y = \frac{1}{3}x^2 + \frac{c}{x}$.

B3.

$$AB = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & -1 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 1 \\ 4 & -2 & -2 \\ -3 & 2 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} = 2I$$

(i) $$A^{-1} = \frac{1}{2}B$$

$$= \frac{1}{2}\begin{pmatrix} 1 & 0 & 1 \\ 4 & -2 & -2 \\ -3 & 2 & 1 \end{pmatrix}$$

(ii) $$A^2B = A.AB = A.2I = 2A$$

$$= \begin{pmatrix} 2 & 2 & 2 \\ 2 & 4 & 6 \\ 2 & -2 & -2 \end{pmatrix}$$

B4. Let $f(x) = (x + 2)\ln(2 + x)$
then $f(0) = 1 + \ln 2$

$$f'(x) = 1 + \ln(x+2) \qquad f'(0) = 1 + \ln 2$$
$$f''(x) = (2 + x)^{-1} \qquad f''(0) = \tfrac{1}{2}$$
$$f'''(x) = -(2 + x)^{-2} \qquad f'''(0) = -\tfrac{1}{4}$$

$$(x + 2)\ln(2 + x) = 2\ln 2 + (1 + \ln 2)x + \frac{1}{2}\frac{x^2}{2!} - \frac{1}{4}\frac{x^3}{3!} + \ldots$$
$$= 2\ln 2 + (1 + \ln 2)x + \frac{x^2}{4} - \frac{x^3}{24} + \ldots$$

B5.

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} - 3y = 6x - 1$$

AE $m^2 + 2m - 3 = 0 \Rightarrow m = 1$ or $m = -3$

Complementary function $y = Ae^{x} + Be^{-3x}$

Particular integral $y = ax + b$

$$\frac{dy}{dx} = a;\ \frac{d^2y}{dx^2} = 0 \Rightarrow 0 + 2a - 3ax - 3b = 6x - 1$$

$$a = -2;\ b = -1$$

General solution

$$y = Ae^{x} + Be^{-3x} - 2x - 1$$

B6. (a) (i) In parametric form, L_2 can be written as $x = -2s;\ y = -2 - s;\ z = 9 + 2s$
Solving x and y

$$8 - 2t = -2s \qquad \Rightarrow s = t - 4$$
$$-4 + 2t = -2 - s \qquad s = 2 - 2t$$

$$t - 4 = 2 - 2t \Rightarrow t = 2 \text{ and } s = -2$$

From L_1, $z = 3 + t = 5$. From L_2, $z = 9 + 2s = 9 - 4 = 5$.
So the lines intersect and do so at $(4,\ 0,\ 5)$.

(ii) Representing an angle between L_1 and L_2 by θ

$$\cos\theta = \frac{(-2)^2 + 2(-1) + 1(2)}{\sqrt{2^2 + 2^2 + 1^2}\sqrt{2^2 + 1^2 + 2^2}} = \frac{4}{9}$$

(b) (i) Direction of L_2 is $-2\mathbf{i} - \mathbf{j} + 2\mathbf{k}$.
Equation of Π is of the form $-2x - y + 2z = k$.
Using $(1,\ -4,\ 2)$ gives $k = -2 + 4 + 4 = 6$ so an equation is $-2x - y + 2z = 6$.

(ii) Substituting $x = 8 - 2t, y = -4 + 2t, z = 3 + t$ into Π gives

$$-16 + 4t + 4 - 2t + 6 + 2t = 6$$
$$4t = 12$$
$$t = 3$$

The point of intersection is $(2,\ 2,\ 6)$.
